Relativizations of Randomness and Genericity Notions

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Abstract

A set A is a base for Schnorr randomness if it is Turing reducible to a set R which is Schnorr random relative to A, and the notion of a base for weak 1-genericity can be defined similarly. We show that A is a base for Schnorr randomness if and only if A is a base for weak 1-genericity if and only if the halting set K is not Turing reducible to A. Furthermore, we define a set A to be high for Schnorr randomness versus Martin-Löf randomness if and only if every set that is Schnorr random relative to A is also Martin-Löf random unrelativized, and we show that A is high for Schnorr randomness versus Martin-Löf randomness if and only if K is Turing reducible to A. Results concerning highness for other pairs of randomness notions are also presented.

Primary Mathematics Subject Classification: 03D32; Secondary Mathematics Subject Classification: 68Q30.

1 Introduction

Kučera and Terwijn [15] showed that there is a nonrecursive set A such that the notions of Martin-Löf randomness relative to A and unrelativized Martin-Löf randomness coincide. As every set is Turing reducible to a Martin-Löf random set [8, 14], A is also Turing reducible to a set which is Martin-Löf random relative to A. Later, this notion was systematically studied [21, 22] and characterized [10].

These studies have been carried out for several other properties as well. In general,

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a set A is called a base for a relativizable property $\mathcal{M}$ if there is a set $B \geq_T A$ that has the property $\mathcal{M}$ relative to A. For example, it is well known that every set is a base for Kurtz randomness (see Remark 1.4 below). Furthermore, no nonrecursive set is a base for 1-genericity, since it is not Turing reducible to any set which is 1-generic relative to it. In the present work, we investigate the bases for the notions of Schnorr randomness and weak 1-genericity and show that in both cases, the bases are the natural class of sets that are not Turing above the halting set. This solves an open problem of Miller and Nies in the case of Schnorr randomness [20, Question 5.2].

There are several notions of algorithmic randomness [4, 17, 19, 27, 28]. A set A is Martin-Löf random if there is no uniformly r.e. sequence of Σ^0_1 classes such that for every e, the e^{th} class has measure at most 2^{-e} and contains A [18]. A set is Schnorr random if "at most 2^{-e}" is replaced by "exactly 2^{-e}" in the previous definition [27]. Alternatively, we can characterize these notions using martingales. A martingale M is a real-valued function defined on finite binary strings such that $\mathrm{M}(\sigma 0) + \mathrm{M}(\sigma 1) = 2\mathrm{M}(\sigma) \geq 0$ for all σ, and it is recursive (r.e.) if and only if the set $\{(\sigma, q) : \sigma \in \{0,1\}^*, q \in \mathbb{Q}, \mathrm{M}(\sigma) > q\}$ is recursive (r.e.). We say that a martingale M succeeds on A if for every c, there is an n such that $\mathrm{M}(A(0)A(1)\dots A(n)) > c$. The Martin-Löf random sets have been characterized as those on which no r.e. martingale is successful [27]. Similarly, a set is recursively random if no recursive martingale succeeds on this set. The martingale characterization of Schnorr randomness is more involved, and there are several versions. Among these, the following is the most suitable for this paper. The martingale characterization of Kurtz randomness is presented here as well.

Property 1.1. *A set R is Schnorr random if there is no recursive martingale* M *and no recursive bound function f such that there are infinitely many n with*

$$\mathrm{M}(R(0)R(1)\dots R(f(n))) > n.$$

A set R is Kurtz random if there is no recursive martingale M *and no recursive bound function f such that for all n,*

$$\mathrm{M}(R(0)R(1)\dots R(f(n))) > n.$$

These notions can be relativized to an oracle A, so we can obtain the corresponding characterizations for "Schnorr random relative to A" and "Kurtz random relative to A" by quantifying over A-recursive functions f and A-recursive martingales M.

Furthermore, a set is called "strongly random" [26] if it is Martin-Löf random and forms a minimal pair with the halting problem. Although the term "weakly 2-random" is widely used [3, 6, 22], in this paper, we will use "strongly random" instead.

Genericity notions [11, 24, 25] are complementary to randomness notions. While a random set avoids certain kinds of null sets, a generic set will either meet some extension or strongly avoid all extensions of every extension function of a certain type. For instance, a set G is 1-generic if and only if for every partial recursive extension function $f : \{0,1\}^* \to \{0,1\}^*$, either there are n and m such that $G(n)G(n+1)\dots G(m) = f(G(0)G(1)\dots G(n-1))$ ("G meets f") or $f(G(0)G(1)\dots G(n-1))$ is undefined for almost all n ("G strongly avoids f"). Weak 1-genericity is a variant in which this condition is required to hold only for total extension functions. In this paper, we will use the following characterization instead, which will allow us to treat f as dependent only on the length of the input and not on the particular choice of input.

Property 1.2. A set G is weakly 1-generic if and only if for every recursive function $f : \mathbb{N} \to \{0,1\}^*$ there are numbers n and m such that $n \leq m$ and $f(n) = G(n)G(n+1)\dots G(m)$.

This notion can, of course, be relativized to an oracle A by quantifying over all A-recursive functions. A set is said to be (weakly) n-generic if it is (weakly) 1-generic relative to $K^{(n-1)}$.

The notion of bases of randomness is linked to lowness. For example, a set A is low for a relativizable property $\mathcal{M}$ if and only if the sets B that have the property $\mathcal{M}$ unrelativized are precisely those that have the property $\mathcal{M}$ relative to A. The most famous result of this type is that a set is low for Martin-Löf randomness if and only if it is a base for Martin-Löf randomness [4, 21, 22].

In the case of bases of Schnorr randomness, there are some parallels to this result if we consider notions of bases of randomness with respect to truth-table reducibility [7]. In the case of Turing reducibility, though, the class of the sets which are low for Schnorr randomness forms a proper subclass of the class of the bases for Schnorr randomness. However, we can develop a notion of highness for Schnorr randomness versus Martin-Löf randomness. This is the dual of lowness for a pair of randomness notions. The concept of lowness for a pair of randomness notions was introduced by Kjos-Hanssen, Nies and Stephan [12]. A set A is said to be low for a notion $\mathcal{M}$ versus a notion $\mathcal{N}$ if and only if every set which has the property $\mathcal{M}$ also has the property $\mathcal{N}$ relative to A. This notion has also been explicitly studied by many others, including Downey, Nies, Weber and Yu [5], Nies [21], and Greenberg and Miller [9]. We formalize the concept of highness for a pair of randomness notions as follows.

Definition 1.3. A set A is high for a notion $\mathcal{M}$ versus a notion $\mathcal{N}$ if and only if every set which has the property $\mathcal{M}$ relative to A also has the property $\mathcal{N}$ unrelativized.

For example, a set A is high for Schnorr randomness versus Martin-Löf randomness if and only if every set which is Schnorr random relative to A is also Martin-Löf random. Note that every set is high for $\mathcal{M}$ versus $\mathcal{N}$ if and only if $\mathcal{M}$ implies $\mathcal{N}$. For example, every set is high for recursive randomness versus Schnorr randomness as every recursively random set is also Schnorr random. If no set is high for $\mathcal{M}$ versus $\mathcal{N}$, then there is a very strong form of nonimplication which cannot be bridged by relativizing $\mathcal{M}$.

The main result of this paper is that this notion is antithetical to being a base for Schnorr randomness and characterizes the Turing degrees above the halting problem. The following properties are shown to be equivalent to $A \geq_T K$.

- A is not a base for Schnorr randomness; that is, there is no $R \geq_T A$ such that R is Schnorr random relative to A (Theorems 2.1 and 2.2).
- A is high for Schnorr randomness versus Martin-Löf randomness; that is, every set which is Schnorr random relative to A is also Martin-Löf random unrelativized (Theorems 2.1 and 2.2).
- A is not a base for weak 1-genericity; that is, there is no $G \geq_T A$ which is weakly 1-generic relative to A (Theorem 3.1).
- A is high for weak 1-genericity versus 1-genericity; that is, every set which is weakly 1-generic relative to A is also 1-generic unrelativized (Corollary 3.2).
- A is high for 1-genericity versus weak 2-genericity; that is, every set which is 1-generic relative to A is also weakly 2-generic unrelativized (Theorem 3.3).

This suggests that highness is very closely connected to the classical recursion-theoretic notion of computational strength and to Turing completeness in particular in a way that lowness is not. The classes of sets that are low for pairs of randomness notions are, of course, all downward closed in the Turing degrees, but their characterizations are given in terms of traceability properties or K-triviality [6, 22]. Only the trivial class of sets that are low for recursive randomness can be described by reference to a single Turing degree. When we study highness for pairs of randomness notions, however, the class of sets under consideration can often be defined in this way.

We also consider bases of recursive randomness and the notion of highness for recursive randomness. We show that no set which is a base for recursive randomness has PA-complete Turing degree. Furthermore, if $A \leq_T K$ and A does not compute a diagonally nonrecursive function, then A is a base for recursive randomness [10]. The following two partial characterizations of the sets which are high for recursive randomness versus Martin-Löf randomness are proven similarly.

- If A is PA-complete, then A is high for recursive randomness versus Martin-Löf randomness.

- If A is high for recursive randomness versus Martin-Löf randomness, then there is a Martin-Löf random set that is Turing reducible to A.

The results for Kurtz randomness are summarized in the following remark, as they are quite straightforward and mostly known.

Remark 1.4. For every set A there is a A'-recursive sequence $a_0, a_1, a_2, \ldots$ of numbers such that R is Kurtz random whenever it is chosen outside the intervals $I_n = \{x : 2^{a_n} \leq x < 2^{a_n+1}\}$ such that betting according to the universal A-r.e. martingale will not increase one's capital, regardless of the values of R on the intervals I_n. Hence, for all $x \in I_n$, we can define $R(x) = A(n)$. As there are only finitely many $m \notin \{a_0, a_1, a_2, \ldots\}$ such that R is constant on the interval $2^m \leq x < 2^{m+1}$, we can compute the positions of the intervals I_n from R and then compute $A(n)$. As R is Kurtz random relative to A and Turing above A, A is a base for Kurtz randomness [13].

Furthermore, the set R constructed here is neither Schnorr random nor weakly 1-generic. In the case of Schnorr randomness, this follows from the fact that R is constant on all intervals I_n. In the case of weak 1-genericity, this follows from the fact that R is either random or constant on the intervals $\{x : 2^m \leq x < 2^{m+1}\}$ but does not meet any other extension requirement. It follows that there is no set A such that A is high for Kurtz randomness versus Schnorr randomness, recursive randomness, Martin-Löf randomness, weak 1-genericity, 1-genericity or weak 2-genericity.

Tables 1 and 2 contain a summary of this information. All of these results appear in this paper except the characterization of highness for Martin-Löf randomness versus strong randomness, which was given by Barmpalias, Miller and Nies [1].

	Schnorr randomness	Recursive randomness	Martin-Löf randomness	Strong randomness
Kurtz randomness	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Schnorr randomness		$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$
Recursive randomness			$\mathcal{P}$	?
Martin-Löf randomness				$\mathcal{D}$

Table 1: Highness for pairs of randomness notions

In these two tables, the entry in row $\mathcal{M}$ and column $\mathcal{N}$ represents the class $\mathcal{C}$ of sets which are high for $\mathcal{M}$ versus $\mathcal{N}$; that is, $\mathcal{C} = \{A :$ every set R satisfying $\mathcal{M}$ relative to A also satisfies $\mathcal{N}\}$. This class of sets is always one of the following:

- the class $\mathcal{K}$ of all $A \geq_T K$,

- the class $\mathcal{K}'$ of all $A \geq_T K'$,
- the class $\mathcal{H}$ of all sets which are high ($A' \geq_T K'$),
- the partially known class $\mathcal{P}$, or
- the class $\mathcal{D}$ of all A such that there is no K-recursive function f which is diagonally nonrecursive relative to A, that is, which satisfies $\varphi_e^A(e) \neq f(e)$ whenever $\varphi_e^A(e)$ is defined.

Although $\mathcal{P}$ is not completely determined, it is known that it contains every PA-complete set and that every set $A \in \mathcal{P}$ is Turing above some Martin-Löf random set; furthermore, not every Martin-Löf random set is in $\mathcal{P}$.

2 Schnorr Randomness

The following theorem is the basis for several of the results in this paper.

Theorem 2.1. *For every set $A \not\geq_T K$ and every set B, there is a set R such that $B \leq_T R$, R is not recursively random and R is Schnorr random relative to A.*

Proof. First, define a recursive injective enumeration $\langle a_m, b_m \rangle$ of all pairs such that $a_m > 0$ and either $b_m = 0$ or some element below a_m is enumerated into K at stage b_m. The enumeration is chosen such that $b_m \leq m$ for all m. Therefore, for each n, there are at most $n + 2$ indices m with $a_m = n$. The largest of these m satisfies $m \geq c_K(n)$, where $c_K(n) = \min\{s \geq n : \forall m \leq n\,[K_s(m) = K(m)]\}$ is the convergence modulus of K. Partition the integers into intervals I_m of length $3a_m + 1$ such that $\min(I_0) = 0$ and $\min(I_{m+1}) = \max(I_m) + 1$ for every m. Now observe that there is a list $M_0, M_1, M_2, \ldots$ of A-recursive martingales such that the n^{th} martingale in this list has the value 1 on all strings of length shorter than n and that whenever some A-recursive martingale succeeds on some set than some member of the list also succeeds on that set. Now we let $\mathrm{M} = \sum_{n=0,1,2,\ldots} \frac{1}{2^{n+2}} M_n$. Note that M itself is not

	Weak 1-genericity	1-genericity	Weak 2-genericity	2-genericity
Kurtz randomness	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Weak 1-genericity		$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}'$
1-genericity			$\mathcal{K}$	$\mathcal{K}$
Weak 2-genericity				$\mathcal{H}$

Table 2: Highness for pairs of genericity notions

A-recursive and that it also has the initial value 1.

Let $F(x) = \max\{m : a_m = x\}$. Note that F majorizes c_K and that F is K-recursive. Let $f_0, f_1, f_2, \ldots$ be a list of all A-recursive functions. Now let $E = \{x_0, x_1, x_2, \ldots\}$, where

$$x_n = \min\{y : \forall m < n\, [x_m < y \wedge f_m(y) < F(y)]\}.$$

Note that every x_n can be defined, as otherwise $F(y) \leq f_0(y) + f_1(y) + \ldots + f_n(y)$ for almost all y. This would contradict the fact that $K \not\leq_T A$. Using E, we can now define the set R inductively on all intervals I_m as follows.

- If $a_m \notin E$ or if there is $k > m$ with $a_m = a_k$, then choose R on I_m such that R has the value 1 on at least one of the least $2a_m$ elements of I_m and M grows on I_m by at most the factor $4^{a_m}/(4^{a_m} - 1)$.
- Otherwise (that is, if $a_m \in E$ and there is no $k > m$ with $a_m = a_k$), choose $R(\min(I_m) + u) = 0$ for $u \in \{0, 1, \ldots, 2a_m - 1\}$ and choose $R(\min(I_m) + u) = B(u - 2a_m)$ for $u \in \{2a_m, 2a_m + 1, \ldots, 3a_m\}$.

Now we show that R has the desired properties.

First, we show that $B \leq_T R$. To compute $B(n)$, search for the first interval I_m such that $a_m \geq n + 1$ and $R(\min(I_m) + u) = 0$ for all $u \in \{0, 1, \ldots, 2a_m - 1\}$. As E contains a number larger than n, the search will terminate. It can be seen that $B(n) = R(\min(I_m) + 2a_m + n)$.

Now we show that R is not recursively random by constructing a recursive martingale N that succeeds on R as follows. The initial capital of N is set as 2 and for each interval I_m, N invests 4^{-a_m}, which is then bet on R being 0 for the first $2a_m$ members of I_m. If all bets are true, then N doubles the invested capital $2a_m$ times and makes a profit of $2^{2a_m} \cdot 4^{-a_m} - 4^{-a_m} = 1 - 4^{-a_m}$. Otherwise, N loses the invested 4^{-a_m}. On one hand, all potential losses can be bounded by $\sum_m 4^{-a_m} \leq \sum_{n>0} (n + 2) \cdot 4^{-n} = \frac{3}{4} + \frac{4}{16} + \frac{5}{64} + \frac{6}{256} + \ldots < 2$ and therefore the martingale never has the value 0. On the other hand, there are infinitely many intervals I_m such that R is 0 on the least $2a_m$ members, so the profit is at least $3/4$ on these intervals and the value of N goes to infinity on R. Thus N witnesses that R is not recursively random.

Finally, we show that R is Schnorr random relative to A. To see this, consider the following function $\tilde{r}(n)$.

$$\tilde{r}(n) = n \cdot \left(\prod_{m<n} 2^{2^{3m+1}} \right) \cdot \left(\prod_{m>0} \left(\frac{4^m}{4^m - 1} \right)^{m+2} \right)$$

Note that this product converges to a real number $\tilde{r}(n) < \infty$ if and only if $\sum_{m>0}(m + 2) \cdot ((4^m/(4^m - 1) - 1) < \infty$; the latter sum is equal to $\sum_{m>0} \frac{m+2}{4^m - 1}$, which, as the following calculation shows, is bounded by 4.

$$\sum_{m>0}(m+2)\cdot\frac{1}{4^m-1}\leq\sum_{m>0}\frac{2^{m+2}}{4^m}\leq\sum_{m>0}2^{2-m}=4$$

Furthermore, $(\prod_{m>0}(\frac{4^m}{4^m-1})^{m+2})$ is a positive real number. Therefore, the function $\tilde{r}$ has a recursive upper bound r such that $r(n)\in\mathbb{N}$ for all n. Without loss of generality, r is chosen such that $r(0)<r(1)<r(2)<\ldots$ holds.

Assume now that M_k is a total A-recursive martingale and f_k is an A-recursive bound function for r as in Property 1.1 such that, in addition, $n<f_k(n)<f_k(n+1)$ for all n. For almost all n,

$$\mathrm{M}_k(R(0)R(1)\ldots R(f_k(n)))\leq n\cdot\mathrm{M}(R(0)R(1)\ldots R(f_k(n))).$$

Now consider $n>x_0+x_1+\ldots+x_k$. Then for each $u<n$, there is at most one interval I_m such that $u\in E$, $m\leq f_k(n)$ and $F(a_m)=u$; for $u\geq n$ there is no interval I_m satisfying these conditions. On the intervals that satisfy these conditions, the martingale M can increase its capital by at most a factor of 2^{3a_m+1}; on all other intervals I_m below $f_k(n)$, M can increase its capital by at most a factor of $4^{a_m}/(4^{a_m}-1)$. Hence, we can see that

$$\mathrm{M}(R(0)R(1)\ldots R(f_k(n)))\leq\frac{r(n)}{n}.$$

It can be seen from the two previous inequalities that for almost all n,

$$\mathrm{M}_k(R(0)R(1)\ldots R(f_k(n)))\leq r(n)$$

and hence R is not Schnorr random relative to A by Property 1.1. ∎

The next result is based on this construction. The equivalence of the first two conditions solves an open problem of Miller and Nies for the special case of Schnorr randomness [20, Question 5.2].

Theorem 2.2. *The following conditions are equivalent for every set A.*

1. *$A\not\geq_T K$.*
2. *A is a base for Schnorr randomness.*
3. *A is not high for Schnorr randomness versus recursive randomness.*
4. *A is not high for Schnorr randomness versus Martin-Löf randomness.*

Proof. To see that (3.) and (4.) imply (1.), we simply note that if $A \geq_T K$, then every set which is Schnorr random relative to A is already recursively random and Martin-Löf random unrelativized.

Furthermore, if $A \geq_T K$, then A is above a low Martin-Löf random set R. Since there is no set which is Martin-Löf random relative to R above R [10] and every set which is Schnorr random relative to A is also Martin-Löf random relative to R, there is no set which is Schnorr random relative to A above A. Therefore, (2.) implies (1.).

If $A \not\geq_T K$, then by Theorem 2.1, there is a set which is above A, Schnorr random relative to A and not recursively random, so (1.) implies (2.) and (3.). Clearly, it is not Martin-Löf random either, so (1.) implies (4.) as well. ∎

Remark 2.3. It should be noted that this characterization can be extended to strong randomness: It holds that $A \geq_T K$ if and only if A is high for Schnorr randomness versus strong randomness. In contrast to this, Barmpalias, Miller and Nies [1] showed that A is high for Martin-Löf randomness versus strong randomness if and only if there is no K-recursive function which is diagonally nonrecursive relative to A.

3 Genericity

The weakly 1-generic sets are a generalization of the 1-generic sets. Their behaviour with respect to Turing degrees can be characterized easily: A Turing degree contains a weakly 1-generic set if and only if it contains a hyperimmune set [16]. Furthermore, a set is low for weak 1-genericity if and only if it is hyperimmune free and not DNR [29]. We now show that the bases for weak 1-genericity also admit a nice characterization.

Theorem 3.1. *A set A is a base for weak 1-genericity if and only if $A \not\geq_T K$.*

Proof. As mentioned in Property 1.2, it is sufficient to consider extension functions that depend only on the length of the string extended. Let $f_0, f_1, f_2, \ldots$ be a list of all total A-recursive functions from $\mathbb{N}$ to $\{0,1\}^*$ and let c_K be the convergence modulus of K as in the proof of Theorem 2.1.

We first suppose that $A \geq_T K$. Every set G which is weakly 1-generic relative to A is also 1-generic unrelativized. There is no 1-generic set above K, so A is not a base for weak 1-genericity.

Now we suppose that $A \not\geq_T K$ and define a set G via a sequence $a_0, a_1, a_2, \ldots$ starting with $a_0 = 0$ inductively as follows:

- $a_{n+1} = a_n + 2 + c_K(n)$,
- $G(a_n) = K(n)$,

- $G(a_n + 1) = A(n)$, and
- $G(a_n + 2)G(a_n + 3) \ldots G(a_{n+1} - 1)$ is $f_k(a_n + 2)0^{c_K(n)-|f_k(a_n+2)|}$ for the first k not used at a previous stage such that $|f_k(a_n + 2)| \leq c_K(n)$.

As there are infinitely many k that map $a_n + 2$ to the empty string, a corresponding extension can always be found and the process will not terminate.

The set G satisfies $A \leq_T G$ and $K \leq_T G$, as one can compute $A(n)$ and $K(n)$ inductively from G given a_n, then $c_K(n)$ from $K(0)K(1) \ldots K(n)$ and, finally, a_{n+1} from a_n and $c_K(n)$.

Assume now for a contradiction that some f_k is never used in this construction. Let k be its index. Then, from some index n onwards, no $k' < k$ is selected due to the nature of the finite injury construction and hence k does not qualify as it cannot be such an index. In other words, for all $n' \geq n$, $|f_k(a_{n'} + 2)| > c_K(n')$. As we can approximate $c_K(n)$ by $c_{K,s}(n) = \max\{t \leq s : \exists m \leq n\,[t = 0 \text{ or } m \text{ goes into } K \text{ at stage } t]\}$, we can compute for $n' \geq n$ the values

- $c_K(n')$ as $c_{K,f_k(a_{n'})}(n')$ and
- $a_{n'+1}$ as $a_{n'} + 2 + c_{K,f_k(a_{n'})}(n')$.

This gives $K \leq_T A$, which produces a contradiction, so every f_k will be built into the construction of G eventually and G will be weakly 1-generic relative to A. Therefore, there is $G \geq_T A$ such that G is weakly 1-generic relative to A and A is a base for weak 1-genericity. ▮

We can also obtain several easy results concerning highness for pairs of genericity notions. Recall that a set is (weakly) n-generic if it is (weakly) 1-generic relative to $K^{(n-1)}$. The following corollary can be seen immediately from the preceding proof.

Corollary 3.2. *A set A is high for weak 1-genericity versus 1-genericity if and only if $A \geq_T K$.*

The sets A such that every set which is 1-generic relative to A is also weakly 2-generic unrelativized have the same characterization.

Theorem 3.3. *A set A is high for 1-genericity versus weak 2-genericity if and only if $A \geq_T K$.*

Proof. One direction is easy: If $A \geq_T K$, then every set which is 1-generic relative to A is 2-generic by definition. For the other direction, we assume that $A \not\geq_T K$. For a given set G, define $\text{next}_G(n) = \min\{m - n : m \geq n \wedge m \in G\}$. This represents the distance to the next element of G after n. The basic idea of the proof is to show that there is a 1-generic set $G \leq_T A'$ such that $\text{next}_G(n) \leq c_K(n)$ for all n. This set will not be weakly 2-generic because it will not meet the K-recursive extension function $f(n) = 0^{c_K(n)+1}1$. The A'-recursive algorithm to produce G is the following. At stage 0, let $e = 0$, $n = 0$ and $G(0) = 1$. At each subsequent stage, proceed as follows.

1. If there is no extension of $G(0)G(1)\ldots G(n)$ in W_e^A, redefine $e = e + 1$.
2. If there is $\sigma \in \{0,1\}^*$ such that $G(0)G(1)\ldots G(n)\sigma \in W_e^A$ and $|\sigma| \leq c_K(n)$, then take the length-lexicographic first such σ, let $G(n+m+1) = \sigma(m)$ for all $m < |\sigma|$ and redefine $n = n + |\sigma|$ and $e = e + 1$.
3. Let $n = n + 1$ and $G(n) = 1$.

Note that there are infinitely many e with $W_e^A = \{0,1\}^*$, so the algorithm never loops in the first step for infinitely many consecutive stages.

Note that the current value $\tilde{e}$ of the variable e is only abandoned if the corresponding value $\tilde{n}$ of n is such that either $G(0)G(1)\ldots G(\tilde{n})$ has no extension in $W_{\tilde{e}}^A$ or after the first $\tilde{n}$ bits, G takes the values of a selected string $\tilde{\sigma}$ such that $G(0)G(1)\ldots G(\tilde{n})\tilde{\sigma}$ is in $W_{\tilde{e}}^A$. Furthermore, there is no single $\tilde{e}$ such that the variable e equals $\tilde{e}$ from some point on, as that would mean that, for the corresponding value $\tilde{n}$, the extension of $G(0)G(1)\ldots G(\tilde{n})1^m$ in $W_{\tilde{e}}^A$ found first (relative to A) always has a length greater than $c_K(\tilde{n}+m)$. This would imply that $K \leq_T A$, contradicting our assumption about A. Therefore, every possible value $\tilde{e}$ of e is eventually taken and eventually abandoned, and G is 1-generic relative to A. It can be seen from the construction that every $\tilde{\sigma}$ added after $\tilde{n}$ has length at most $c_K(\tilde{n})$ and is followed by a 1, so $\text{next}_G(\tilde{n}) \leq c_K(\tilde{n})$ for all $\tilde{n}$. This completes the proof. ∎

On one hand, if G is 1-generic relative to K, then G is already 2-generic. On the other hand, if $A \not\geq_T K$, then the preceding result shows that there is a set G which is 1-generic relative to A but not weakly 2-generic. Obviously, G is not 2-generic in this case. This gives us the following corollary.

Corollary 3.4. *A set A is high for 1-genericity versus 2-genericity if and only if $A \geq_T K$.*

Note that G is weakly 2-generic relative to A if and only if it is weakly 1-generic relative to A'. Furthermore, G is 2-generic if and only if G is 1-generic relative to K. This means that we can relativize Corollary 3.2 to see that whenever $A' \not\geq_T K'$, there is a set G which is weakly 1-generic relative to A' but not 1-generic relative to K; it follows that G is weakly 2-generic relative to A but not 2-generic. On the other hand, if $A' \geq_T K'$, then every set which is weakly 2-generic relative to A is also weakly 1-generic relative to A', weakly 1-generic relative to K', weakly 3-generic and 2-generic. This gives us the following corollary.

Corollary 3.5. *A set A is high for weak 2-genericity versus 2-genericity if and only if A is high; that is, if and only if $A' \geq_T K'$.*

Now consider a set A that is high for weak 1-genericity versus 2-genericity. As every 2-generic set is 1-generic, Corollary 3.2 lets us see that $A \geq_T K$. Hence $A \equiv_T B'$ for some B by the Jump Inversion Theorem and the sets which are weakly 1-generic relative to A are precisely those which are weakly 2-generic relative to B. It follows from Corollary 3.5 that $B' \geq_T K'$, so $A \geq_T K'$.

Conversely, consider any $A \geq_T K'$. Every set which is weakly 1-generic relative to A is also weakly 3-generic unrelativized, so A is high for weak 1-genericity versus 2-genericity. This is summarized in the following corollary.

Corollary 3.6. *A set A is high for weak 1-genericity versus 2-genericity if and only if $A \geq_T K'$.*

4 Recursive Randomness

Theorem 2.2 states that a set A is high for Schnorr randomness versus recursive randomness if and only if $A \geq_T K$. Furthermore, if A is PA-complete, we can get the following result.

Proposition 4.1. *If A is PA-complete, then A is high for recursive randomness versus Martin-Löf randomness.*

Proof. Let M be an r.e. martingale which succeeds on every nonrandom set and which satisfies

$$\sum_{\sigma \in \{0,1\}^n} \mathrm{M}(\sigma) = r \cdot 2^n$$

for some real $r < \frac{1}{2}$ and for all n. Now there is a martingale N satisfying the following conditions:

- $\mathrm{M}(\sigma) < \mathrm{N}(\sigma) \leq 2^{|\sigma|}$ for all σ,
- $\sum_{\sigma \in \{0,1\}^n} \mathrm{N}(\sigma) = 2^n$ for all n, and
- $\mathrm{N}(\sigma) \in \{0, 2^{-|\sigma|-2}, 2 \cdot 2^{-|\sigma|-2}, 3 \cdot 2^{-|\sigma|-2}, \ldots, 2^{|\sigma|}\}$ for all σ.

In order to see that such an N exists, we define a sequence of martingales $\mathrm{M}_0, \mathrm{M}_1, \mathrm{M}_2, \ldots$ such that $\mathrm{M}_0(\sigma) = \frac{1}{2} - r$ for all σ and, for $n > 0$, $\mathrm{M}_n(\sigma)$ equals 2^{-n-1} when $|\sigma| < n$ and bets such that for $\sigma \in \{0,1\}^n$, the sum $\mathrm{M}(\sigma) + \sum_{m \leq n} \mathrm{M}_m(\sigma)$ is a multiple of 2^{-n-2}. To guarantee this, M_n only bets on the n^{th} bit in a way that ensures that the bets of $\mathrm{M} + \sum_{m<n} \mathrm{M}_m$ on this bit get rounded to a multiple of 2^{-n-2}. After that, M_n abstains from betting. Then we define $\mathrm{M} + \sum_n \mathrm{M}_n$. As a result, for $\sigma \in \{0,1\}^n$,

$$\mathrm{N}(\sigma) = \mathrm{M}(\sigma) + \sum_{m \leq n} \mathrm{M}_n(\sigma) + \sum_{m>n} 2^{-m-1}$$

and, as both sums $\mathrm{M}(\sigma) + \sum_{m \leq n} \mathrm{M}_n(\sigma)$ and $\sum_{m>n} 2^{-m-1}$ are multiples of 2^{-n-2}, so is N.

The resulting martingale N takes only one of finitely many possible values on every input σ, namely one of the multiples of $2^{-|\sigma|-2}$ which is at least 0 and at most $2^{|\sigma|}$. Furthermore, N majorizes M and thus succeeds on the same sets.

The class of all N with the three properties above forms a Π^0_1 class, as $\mathrm{M}(\sigma)$ can be approximated from below and we find out eventually if $q < \mathrm{M}(\sigma)$ for some rational q. Note that the values of M are not rationals, as otherwise there would be nonrandom sets on which M does not succeed. So, as A is PA-complete, N can be chosen to be an A-recursive martingale taking one of a given finite collection of rational values on each input.

Now we can see that if R is recursively random relative to A, then N is not successful on R, so M does not succeed on R either and R is Martin-Löf random. ∎

Now we ask what can be said about the other direction. Indeed, the above result is not known to be a characterization and the Turing degrees of many Martin-Löf random sets are not PA-complete. Therefore, the next result is not a full characterization.

Theorem 4.2. *If A is high for recursive randomness versus Martin-Löf randomness, then there is a Martin-Löf random set $R \leq_T A$.*

Proof. Let A be a set that does not bound any Martin-Löf random set. We will show that A is not high for recursive randomness versus Martin-Löf randomness. This will be done by constructing a function $F \leq_T A'$ such that no Martin-Löf random set is Turing reducible to $A \oplus F$ and $A \oplus F$ has high Turing degree relative to A (that is, $A'' \leq_T (A \oplus F)'$). Then there will be a set $Q \leq_T A \oplus F$ which is recursively random relative to A [23]. As Q will not be Martin-Löf random, it follows automatically that A is not high for recursive randomness versus Martin-Löf randomness.

In order to code highness in F, we consider an A'-recursive injective enumeration $e_0, e_1, e_2, \ldots$ of all indices of strictly partial A-recursive functions such that for all k, there is an $x \leq k$ with $\varphi^A_{e_k}(x)$ undefined. Furthermore, M denotes the universal r.e.

martingale and, for computations relative to a partial function ψ as an oracle, $\varphi_e^\psi(x)$ is undefined whenever the computation asks for some value of ψ outside the domain of ψ.

The function $F : \mathbb{N} \to \mathbb{N}$ is defined by stepwise extensions, starting with $\sigma_0 = 000$ and by building $\sigma_{n+1} = \sigma_n e_n \tau_{n,0} \tau_{n,1} \ldots \tau_{n,n}$ where each string $\tau_{n,m}$ is chosen from $\{e_n, e_n + 1, e_n + 2, \ldots\}^*$ such that $\eta = \sigma_n e_n \tau_{n,0} \tau_{n,1} \ldots \tau_{n,m}$ satisfies one of the following two conditions for all $m \leq n$.

- $\mathrm{M}(\rho) > n$ for some $\rho \preceq \varphi_{m,|\eta|}^{A \oplus \eta}$.
- There is $x < |\eta|$ with $\neg(\varphi_m^{A \oplus \eta G}(x){\downarrow} \in \{0, 1\})$ for all $G \in \{e_n, e_n + 1, e_n + 2, \ldots\}^\infty$.

To verify the construction, we first show that such an extension can always be found. Let n and m be given, and let $\vartheta = \sigma_n e_n \tau_{n,0} \tau_{n,1} \ldots \tau_{n,m-1}$. Given A, one can iteratively search the strings $\gamma_0, \gamma_1, \gamma_2, \ldots \in \{e_n, e_n + 1, e_n + 2, \ldots\}^*$ such that

$$E(k) = \varphi_m^{A \oplus \vartheta \gamma_0 \gamma_1 \gamma_2 \ldots \gamma_k}(k){\downarrow} \in \{0, 1\}$$

for each k where γ_k is found. If there is no such γ_k, we append the empty string.

If this can be done for all k, then $E \leq_T A$ and E is not Martin-Löf random. Therefore, there is a k such that $\mathrm{M}(E(0)E(1)E(2) \ldots E(k)) > n$, and we can choose $\tau_{n,m} = \gamma_0 \gamma_1 \gamma_2 \ldots \gamma_k$ and satisfy the first condition in the definition of $\tau_{n,m}$.

On the other hand, if this construction goes through up to some k but not beyond, it is impossible to define $E(k + 1)$ with a value in $\{0, 1\}$. In this case, we choose $\tau_{n,m} = \gamma_0 \gamma_1 \gamma_2 \ldots \gamma_k$ for this k and satisfy the second condition in the definition of $\tau_{n,m}$.

Second, we show that the resulting F is such that $A \oplus F$ has a Turing degree which is high relative to A. Given m, one can, relative to F', find the largest k such that $F(m) \leq k$. It follows from the construction that whenever there is an n such that $e_n = m$, then $n \leq |\sigma_n| \leq k$. Hence one can check whether there is an $n \leq k$ such that $e_n = m$ relative to A' (since for all k, there is an $x \leq k$ such that $\varphi_{e_k}^A(x)$ is undefined, we have a bound on our search). Then the overall algorithm is recursive in $A' \oplus F'$ and thus $A'' \leq_T (A \oplus F)'$. In other words, $A \oplus F$ has high Turing degree relative to A.

Third, we show that there is no Martin-Löf random set recursive in $A \oplus F$. To see this, consider any m such that $\varphi_m^{A \oplus F}$ is total and $\{0, 1\}$-valued. Furthermore, consider the infinitely many $n \geq m$ satisfying $\forall k > n\, [e_n < e_k]$. These n must exist, as $e_0, e_1, e_2, \ldots$ is an injective enumeration of an infinite set. The extension $\tau_{n,m}$ cannot be selected as in the second item above, as then $\varphi_m^{A \oplus F}$ would either be partial or not $\{0, 1\}$-valued. Therefore, the extension $\tau_{n,m}$ is chosen according to the first condition and there is some $\rho \preceq \varphi_m^{A \oplus \eta}$ such that $\mathrm{M}(\rho) > n$ for $\eta = \sigma_n e_n \tau_{n,0} \tau_{n,1} \ldots \tau_{n,m}$. As ρ is a prefix of $\varphi_m^{A \oplus F}$, it follows that M succeeds on $\varphi_m^{A \oplus F}$ and $\varphi_m^{A \oplus F}$ is not Martin-Löf random. This completes the proof. ∎

The next result shows that the above result is not optimal.

Theorem 4.3. *There is a Martin-Löf random set A which is not high for recursive randomness versus Martin-Löf randomness.*

Proof. Cholak, Greenberg and Miller [2] constructed an r.e. set $B <_T K$ and a function $f \leq_T B$ such that for a subclass of $\{0,1\}^\infty$ of measure 1, every function recursive relative to a member of this class is dominated by f. The set B is then said to be uniformly almost everywhere dominating.

A recursive martingale can be given by a function φ_e which computes a rational number q between 0 and 2 on every input σ that says how to bet on the next bit. In other words, the capital at $\sigma 1$ is q times the old capital and the capital at $\sigma 0$ is $(2-q)$ times the old capital. It is known that the notion of recursive randomness (relative to some oracle) is the same whether real-valued or rational-valued martingales are used [27], so we can describe the martingales using the functions φ_e. In the case of an oracle E, we consider the function φ_e^E instead.

Now we produce an B-recursive martingale M (the superscript B is omitted here and from now on in order to keep notation simple) which follows the following strategy: For each oracle E and each index e, M_e^E computes the capital $\mathrm{M}_e^E(\sigma)$ using the base case $\mathrm{M}_e^E(\sigma) = 1$ when $|\sigma| \leq e$. If $|\sigma| \geq e$ and $a \in \{0,1\}$, then we define M_e^E inductively using the following formula:

$$\mathrm{M}_e^E(\sigma a) = \begin{cases} q \cdot \mathrm{M}_e^E(\sigma) & \text{if } a = 1 \text{ and } q = \varphi_e^E(\sigma) \text{ is in } \mathbb{Q},\ 0 \leq q \leq 2 \\ & \text{and } q \text{ is computed with time and use } f(|\sigma|); \\ (2-q) \cdot \mathrm{M}_e^E(\sigma) & \text{if } a = 0 \text{ and } q = \varphi_e^E(\sigma) \text{ is in } \mathbb{Q},\ 0 \leq q \leq 2 \\ & \text{and } q \text{ is computed with time and use } f(|\sigma|); \\ \mathrm{M}(\sigma) & \text{otherwise.} \end{cases}$$

The martingale M is defined as

$$\mathrm{M}(\sigma) = \sum_{e=0,1,2,\dots} 2^{-e-1} \sum_{\tau, |\tau| = f(|\sigma|)} \mathrm{M}_e^\tau(\sigma)$$

and M is B-recursive since $\mathrm{M}_e^E(\sigma)$ can be computed from the first $f(|\sigma|)$ bits of E for each σ and E and, furthermore, $\mathrm{M}_e^E(\sigma)$ can only differ from 1 when $e \leq |\sigma|$.

We can now choose a B-recursive set R on which M is not successful; M does not make any profit on this set and $\mathrm{M}(B(0)B(1)\dots B(n)) \leq 1$ for all n. The set R is not Martin-Löf random as B is r.e. and Turing incomplete [14].

Now we show that R is recursively random relative to every member A of a class $\mathcal{A}$ of measure 1. Assume for a contradiction that this is not the case. Then there must be a fixed martingale N such that N^A is A-recursive and N^A succeeds on R for a set of oracles A which does not have measure 0. Using arguments given by Mihailović [19] as well as Franklin and Stephan [7], we can assume that N has the savings property and that $\mathrm{N}^A(\sigma\tau) \geq \mathrm{N}^A(\sigma) - 2$ for all $\sigma, \tau \in \{0,1\}^*$. The class

$$\mathcal{A} = \{A : \mathrm{N}^A \text{ is total and } \forall c \exists n \, [\mathrm{N}^A(R(0)R(1)\ldots R(n)) > c]\}$$

is measurable and hence has positive measure. Therefore, if f dominates all A-recursive functions, then there is a constant r_A such that $\mathrm{N}^A(\sigma) \leq r_A \cdot \mathrm{M}_e^A(\sigma)$ for all σ, as M_e^A is computed using the function $\varphi_e^A(\sigma)$ for almost all σ and $f \leq_T A$ bounds the use for M_e^A. Since we can require that $r_A \in \mathbb{N}$, there are only countably many choices for each A and so there must be one fixed constant r and some $\epsilon > 0$ such that the class

$$\mathcal{B} = \{A \in \mathcal{A} : \forall \sigma \in \{0,1\}^* \, [\mathrm{N}^A(\sigma) \leq r \cdot \mathrm{M}_e^A(\sigma)]\}$$

has measure ϵ. Due to the savings property of N, there is a function g such that the measure of each class

$$\mathcal{C}_n = \{A \in \mathcal{B} : \mathrm{N}^A(R(0)R(1)\ldots R(g(n))) > n+1\}$$

is at least $\epsilon \cdot \frac{n}{n+1}$, since $g(n)$ is simply the first m such that for sufficiently many members of $\mathcal{B}$, M has already reached a value above $n+3$ after processing $R(0)R(1)\ldots R(m)$ and, is therefore still above $n+1$ due to the savings property. It follows for all n that

$$\mathrm{M}(R(0)R(1)\ldots R(g(n))) \geq \epsilon \cdot \tfrac{n}{n+1} \cdot 2^{-e-1} \cdot \tfrac{1}{r} \cdot (n+1) = \epsilon \cdot 2^{-e-1} \cdot \tfrac{1}{r} \cdot n,$$

which contradicts the fact that M does not succeed on R. Hence R is recursively random relative to all members of a class of measure one, and this class must have a member that is Martin-Löf random. In other words, R witnesses that there is a Martin-Löf random set A which is not high for recursive randomness versus Martin-Löf randomness. ∎

We note that this proof actually shows not just that there is a Martin-Löf random set that is not high for recursive randomness versus Martin-Löf randomness, but that this highness class is actually null.

Acknowledgments. The authors would like to thank Joseph Miller for his correspondence and useful discussions and the anonymous referee for his or her helpful comments.

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