Schnorr trivial sets and truth-table reducibility

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Abstract

We give several characterizations of Schnorr trivial sets, including a new lowness notion for Schnorr triviality based on truth-table reducibility. These characterizations allow us to see not only that some natural classes of sets, including maximal sets, are composed entirely of Schnorr trivials, but also that the Schnorr trivial sets form an ideal in the truth-table degrees but not the weak truth-table degrees. This answers a question of Downey, Griffiths and LaForte.

1 Introduction

One of the major achievements in the study of Martin-Löf randomness is the discovery that the following statements about a set A are equivalent [9, 15].

- A is low for Martin-Löf randomness; that is, every set that is Martin-Löf random unrelativized is also Martin-Löf random relative to A.
- A is low for prefix-free Kolmogorov complexity; that is, the prefix-free Kolmogorov complexity relative to A differs from the unrelativized version only by a constant.
- A is trivial for prefix-free Kolmogorov complexity; that is, the complexity of $A{\upharpoonright}n$ is less than or equal to the complexity of 0^n plus a constant.

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- A is a basis for Martin-Löf randomness; that is, A is Turing reducible to a set which is Martin-Löf random relative to A.

Nies also showed in [15] that these sets form an ideal with respect to Turing reducibility.

Schnorr felt that Martin-Löf's definition [12] was too restrictive, so he introduced an alternative definition of randomness [21, 22]. This definition gives rise to a class of sets with very different properties. For example, Martin-Löf random sets do not exist in any incomplete r.e. Turing degree [1], while Schnorr random sets exist in every high Turing degree [16].

A triviality notion can be defined for Schnorr randomness as well as for Martin-Löf randomness [3, 5, 6]. The triviality notions for Martin-Löf and Schnorr randomness are even more different than the original randomness notions. For instance, the Schnorr trivial sets are not closed downward under Turing reductions [3]. Furthermore, Schnorr triviality does not coincide with lowness for Schnorr randomness when the latter is defined using Turing reducibility. In fact, only the Schnorr trivial sets of hyperimmune-free Turing degree are low for Schnorr randomness [7]. Once again, these notions do not behave in the way we would expect with regard to Turing reducibility.

In this paper, we consider whether there is a way in which the lowness, triviality, and basis notions for Schnorr randomness can be made to coincide. To this end, we consider lowness for Schnorr randomness from the perspective of truth-table reducibility rather than Turing reducibility. In Section 2, we define the notion of "truth-table lowness for Schnorr randomness" by giving a notion of Schnorr randomness relative to A that is sensitive to truth-table reducibility. In Section 3, we prove that Schnorr triviality and truth-table lowness for Schnorr randomness coincide, along with several other characterizations. In Section 4, we use these results to show that all maximal sets are Schnorr trivial and that this observation cannot be generalized to r-maximal or cohesive sets. In Section 5, a proof is given for the result from Franklin's thesis [5] that the Schnorr trivial sets form a truth-table ideal, which answers a question of Downey, Griffiths and LaForte [3]. Furthermore, we show that Schnorr trivial sets are not closed under weaker reducibilities than truth-table reducibility, such as weak truth-table reducibility. This suggests that truth-table reducibility is the most appropriate reducibility for the study of Schnorr triviality. Finally, in Section 6, we show that the Schnorr trivial sets cannot be characterized as the bases of Schnorr randomness with respect to truth-table reducibility.

We refer the reader to Odifreddi [17, 18], Rogers [19] and Soare [23] for an overview of recursion theory and to Li and Vitányi [11], Schnorr [21] and Downey, Hirschfeldt, Nies and Terwijn [4] for an overview of algorithmic randomness. However, before we begin, we recall the martingale characterizations of Schnorr randomness and recursive randomness from [21] as well as a definition of Schnorr triviality that is easily seen to be equivalent to the original in [3]. A martingale is simply a function $d : \{0,1\}^* \to \mathbb{R}^{\geq 0}$ such that $d(\sigma) = \frac{d(\sigma^\frown 0)+d(\sigma^\frown 1)}{2}$ and a recursive martingale d is a martingale whose values are uniformly recursive reals. We will refer to unbounded, nondecreasing recursive functions as order functions throughout the paper.

Definition 1.1. *A set R is Schnorr random if there is no recursive martingale d and no order function f such that $d(R{\restriction}n) \geq f(n)$ for infinitely many n.*

Definition 1.2. *A set R is recursively random if there is no recursive martingale such that for every m, there is an n such that $d(R{\restriction}n) \geq m$.*

Definition 1.3. *A set A is Schnorr trivial if for every prefix-free machine M such that $\sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|} = 1$, there are a prefix-free machine N and a constant c such that the following two conditions hold.*

1. $\sum_{\sigma \in \mathrm{dom}(N)} 2^{-|\sigma|} = 1$.
2. *For all n and all $\sigma \in \mathrm{dom}(M)$ such that $M(\sigma) = 0^n$, there is a $\tau \in \mathrm{dom}(N)$ such that $N(\tau) = A{\restriction}n$ and $|\tau| \leq |\sigma| + c$.*

2 Relativizing Schnorr randomness

Before we consider Schnorr randomness with respect to truth-table reducibility, we provide some alternate characterizations of the sets that are not Schnorr random and discuss which of these notions is most suitable for relativization.

Proposition 2.1. *The following statements are equivalent for a set R.*

1. *There is a recursive martingale d and a recursive function f such that $d(R{\restriction} f(n)) \geq n$ for infinitely many n.*
2. *There is a recursive martingale $\dot{d}$ and a recursive function $\dot{f}$ such that $\dot{d}(\sigma\tau) + 2 \geq \dot{d}(\sigma)$ for all $\sigma, \tau \in \{0,1\}^*$ and $\dot{d}(R{\restriction}\dot{f}(n)) \geq n$ for infinitely many n.*
3. *The set R is not Schnorr random; that is, there is a recursive martingale $\ddot{d}$ and an order function $\ddot{f}$ such that $\ddot{d}(R{\restriction}n) \geq \ddot{f}(n)$ for infinitely many n.*

Proof. To see that the first two are equivalent, it is enough to show that the first implies the second. We assume without loss of generality that d never takes a power of 2 as its value and that the initial value of d is a recursive real number strictly between

0 and 1. We begin by defining a sequence $d_0, d_1, \ldots$ of recursive martingales with the same initial value as d that satisfy the following rule for $a \in \{0,1\}$.

$$d_k(\sigma^\frown a) = \begin{cases} d(\sigma^\frown a) & \text{if } d_k(\sigma) < 2^k \\ d_k(\sigma) & \text{if } d_k(\sigma) \geq 2^k \end{cases}$$

Since $d(\sigma)$ is never a power of 2 and has an initial value between 0 and 1, this case distinction will be recursive. Now we let

$$\dot{d}(\sigma) = \sum_{k \geq 0} 2^{-k} \cdot d_k(\sigma)$$

for all $\sigma \in \{0,1\}^*$. Furthermore, we define $\dot{f}(n) = \max\{f(m) : m \leq 2^{n+1}\}$. This translation of the bounds is based on the observation that

$$\exists^\infty n \, \exists m \in \{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\} \, [d(R{\restriction}f(m)) \geq m].$$

By induction over the definition of d_k, we can show that for infinitely many n, there is some $m < 2^{n+1}$ such that for all $k \leq n$, the inequalities $d_k(R{\restriction}f(m)) \geq 2^k$ and $d_k(R{\restriction}\dot{f}(n)) \geq 2^k$ hold. It follows that

$$\dot{d}(R{\restriction}\dot{f}(n)) \geq \sum_{k \leq n} 2^{-k} d_k(R{\restriction}\dot{f}(n)) \geq \sum_{k \leq n} 2^{-k} \cdot 2^k \geq n$$

for all but finitely many n. Now we only need to show that

$$\forall \sigma, \tau \in \{0,1\}^* \, [\dot{d}(\sigma^\frown \tau) + 2 \geq \dot{d}(\sigma)].$$

Given any σ and τ, we choose n to be the maximal integer such that there is a prefix θ of σ with $2^n \leq d(\theta)$. For $k > n$, the martingale d_k will behave like d, while for $k \leq n$, the martingale d_k is constant above σ. Then the following two equations hold.

$$\begin{aligned} \dot{d}(\sigma) &= \sum_{k \leq n} 2^{-k} d_k(\sigma) + \sum_{k > n} 2^{-k} \cdot d(\sigma) \\ \dot{d}(\sigma^\frown \tau) &= \sum_{k \leq n} 2^{-k} d_k(\sigma^\frown \tau) + \sum_{k > n} 2^{-k} \cdot d(\sigma^\frown \tau) \end{aligned}$$

The second term in the formula for $\dot{d}(\sigma)$ is bounded by 2, and this is the only part that may change when we consider $\sigma^\frown \tau$ instead of σ. Since n was chosen such that $\sum_{k \leq n} 2^{-k} d_k(\sigma^\frown \tau) = \sum_{k \leq n} 2^{-k} d_k(\sigma)$ and $\sum_{k > n} 2^{-k} d_k(\sigma) \leq 2$, the condition

$$\forall \sigma, \tau \in \{0,1\}^* \, [\dot{d}(\sigma^\frown \tau) + 2 \geq \dot{d}(\sigma)]$$

is satisfied. Therefore, the first statement implies the second.

To prove that the second statement implies the third, we first show that $\dot{f}$ can be assumed to be strictly increasing. Given a recursive function $\dot{g}$, one can define a function $\dot{f}$ recursively by the following two equations.

$$\begin{aligned} \dot{f}(0) &= \dot{g}(0) + \dot{g}(1) + \dot{g}(2) \\ \dot{f}(n+1) &= \dot{g}(n+3) + \dot{f}(n) + 1 \end{aligned}$$

There are infinitely many n such that $\dot{d}(R{\restriction}\dot{g}(n+2)) \geq n+2$, so, since $\dot{f}(n) \geq \dot{g}(n+2)$ for all n, it follows that $\dot{d}(R{\restriction}\dot{f}(n)) \geq n$. Now let $\ddot{f}(n) = \max\{m : m = 0 \vee \dot{f}(m) \leq n\}$ for all n. As there are infinitely many m such that

$$\dot{d}(R{\restriction}\dot{f}(m)) \geq m,$$

we can take the value $n = \dot{f}(m)$ for these m and see that

$$\dot{d}(R{\restriction}n) \geq \ddot{f}(n).$$

Therefore, we can set $\ddot{d} = \dot{d}$ to see that the third statement follows from the second.

To see that the second statement follows from the third, assume that $\ddot{d}$ and $\ddot{f}$ are given and that $\dot{d}$ is built from $\ddot{d}$ as $\dot{d}$ was built from d above. Furthermore, for every m, let

$$f(m) = \min\{n : \ddot{f}(n) > 2^{m+4}\}$$

and note that f will always be defined because $\ddot{f}$ is unbounded. Furthermore, for all $\sigma, \tau \in \{0,1\}^*$, $\dot{d}(\sigma\frown\tau) \geq m$ whenever $\ddot{d}(\sigma) \geq 2^m$. For infinitely many m there is an $n \in \{f(m), f(m)+1, \ldots, f(m+1)-1\}$ such that $\ddot{d}(R{\restriction}n) \geq \ddot{f}(n)$. It follows that $\dot{d}(R{\restriction}n\frown\tau) \geq m$ for all τ and, in particular, that $\dot{d}(R{\restriction}\dot{f}(m)) \geq m$. ∎

This characterization of sets that are not Schnorr random will help us to establish a link between the sets that are Schnorr random relative to A in the context of truth-table reducibility and the Schnorr triviality of A. First, we show that the third characterization in Proposition 2.1 is not suitable for relativization.

Theorem 2.2. *There is a Schnorr random set $R \equiv_T K$ and a recursive martingale d such that for every $A \geq_T K$, there is a nondecreasing unbounded function $lb \leq_{tt} A$ such that $d(R{\restriction}n) \geq lb(n)$ for all n.*

Proof. Nies, Stephan, and Terwijn [16] showed that there is a set $R \equiv_T K$ which is Schnorr random but not recursively random. Therefore, there is a recursive martingale d which succeeds on R, although not with the bounds required for Schnorr randomness. In Proposition 2.1, we showed that we can replace d by another recursive martingale $\dot{d}$ such that $\dot{d}(R{\restriction}m) \geq \dot{d}(R{\restriction}n) - 2$ for all n and all $m > n$. Since our original

d was truth-table computable from K and this transformation preserves truth-table reducibility, we can see that $\dot{d} \leq_{tt} K$. Now let $lb(n)$ be the largest natural number k such that either $k = 0$ or there is a number $m \leq n$ for which $\dot{d}(R \upharpoonright m) \geq k+2$ and $R \upharpoonright m$ can be computed from A in no more than m steps. Since all the numbers involved are bounded, this last statement can be evaluated in the context of a truth-table reduction. It is easy to see that $\dot{d}(R \upharpoonright n) \geq lb(n)$ for all n. Furthermore, lb is nondecreasing by definition. It is also easy to see that lb is unbounded, as $\dot{d}$ takes arbitrarily large values on R. ∎

This indicates that the following definition is a more suitable candidate for a notion of relativized Schnorr randomness in the context of truth-table reducibility.

Definition 2.3. A set R is truth-table Schnorr random relative to A if and only if there is no martingale $d \leq_{tt} A$ and no recursive bound function b such that

$$\exists^{\infty} n \, [d(R \upharpoonright b(n)) \geq n].$$

A set A is truth-table low for Schnorr randomness if every Schnorr random set R is truth-table Schnorr random relative to A.

Remark 2.4. Technically, we should consider all bound functions b that are truth-table reducible to A instead of only recursive bound functions, but this is not necessary. The proof of the second statement from the first in Proposition 2.1 preserves truth-table reducibility relative to a given set, so we may assume this without loss of generality. If a bound function is computed via a truth-table reduction, there are only finitely many choices for its value at any given n if the oracle is unknown. Hence, given a bound $\tilde{b} \leq_{tt} A$, we can obtain a new recursive bound $b(n)$ by taking the maximum of all possible values of $\tilde{b}(4n+4)$. There are infinitely many n for which there is an $m \in \{4n, 4n+1, 4n+2, 4n+3\}$ such that $d(R \upharpoonright \tilde{b}(m)) \geq m$, so $d(R \upharpoonright b(n)) \geq n$ for these n as well. Thus, d and the new recursive bound b witness that R is not truth-table Schnorr random relative to A.

3 Characterizing Schnorr triviality

In this section, we will give several equivalent characterizations of Schnorr triviality. The following theorem, which states that every set A must either truth-table compute a martingale that succeeds on some recursively random set in the sense of Schnorr or be "captured" in a particular way by a recursive function, is an important preliminary result. We will show later that the second statement in the following theorem is equivalent to Schnorr triviality.

Theorem 3.1. *For every set A, exactly one of the following two statements holds.*

1. *There is a recursively random set R that is not truth-table Schnorr random relative to A.*

2. *For every recursive function u, there is a recursive function g such that for almost all n,*
$$A{\upharpoonright}u(n) \in \{g(0), g(1), \ldots, g(16^n)\}.$$

Proof. We will show that for every recursive function u, either the function g exists or a set $R \leq_T A''$ can be constructed which is recursively random but not truth-table Schnorr random relative to A. Note that it is sufficient to consider only strictly increasing functions u.

Construction. Let $d_1, d_2, \ldots$ be a listing of all the recursive martingales that only assume positive values. It is clear that it is enough to diagonalize against these martingales. We will incorporate each of these martingales into the martingale that we are constructing, $\dot{d}$, at some level c_k.

For each n, we define $J_n = \{0,1\}^{u(n)+1}$ and partition the natural numbers into a sequence of intervals I_σ, where $\sigma \in \bigcup_n J_n$, such that if $\sigma \in J_n$, then I_σ has $2n$ elements. Now we inductively define our set R and martingale $\dot{d}$ as follows.

At the beginning, we set $k = 0$ and define the initial value of $\dot{d}$ to be 1. For each successive I_σ, we let n be the unique number such that $\sigma \in J_n$, let $m = \min(I_\sigma)$ and let k be the largest index of the martingales incorporated into $\dot{d}$ so far. Note that $c_1, c_2, \ldots, c_k$ are already defined and assume values between 0 and m. We will extend the definition of the martingale $\dot{d}$ on all $\tau \in \{0,1\}^*$ with $m+1 \leq |\tau| \leq m+2n$ such that the equation
$$\dot{d}(\tau) = 2^{-k} + \sum_{1 \leq j \leq k} 2^{-j} \cdot \tfrac{d_j(\tau)}{d_j(\tau{\upharpoonright}c_j)}$$
holds. If $k = 0$, this just means that $\dot{d}(\tau) = 1$ for all such τ. We can see that this equation will hold at each stage of the construction.

Now that we have extended $\dot{d}$, we check to see which of the following three cases holds for σ.

Case 1: $\sigma \neq A{\upharpoonright}u(n)$.

Case 2: $\sigma = A{\upharpoonright}u(n)$ and $\dot{d}(R{\upharpoonright}m^\frown 1^{2n}) \leq \dot{d}(R{\upharpoonright}m) + 2^{-n}$.

Case 3: $\sigma = A{\upharpoonright}u(n)$ and $\dot{d}(R{\upharpoonright}m^\frown 1^{2n}) > \dot{d}(R{\upharpoonright}m) + 2^{-n}$.

In Cases 1 and 3, we define R on I_σ to be such that $\dot{d}(R{\upharpoonright}(m+2n))$ is as small as possible. In Case 1, we can see that this will guarantee that
$$\dot{d}(R{\upharpoonright}(m+2n)) \leq \dot{d}(R{\upharpoonright}m).$$

In Case 3, the 2^{-n} gain on the sequence 1^{2n} must be balanced by a loss of at least 8^{-n} on at least one of the $4^n - 1$ other possible extensions of R on I_σ, so

$$\dot{d}(R\restriction(m+2n)) \leq \dot{d}(R\restriction m) - 8^{-n}.$$

We do not increase k in either of these cases.

In Case 2, let $R(m+i) = 1$ for $i \in \{1, 2, \ldots, 2n\}$, and choose $c_{k+1} = m + 2n$ such that

$$\dot{d}(\tau) = 2^{-k-1} + \sum_{1 \leq j \leq k+1} 2^{-j} \cdot \tfrac{d_j(\tau)}{d_j(\tau\restriction c_j)}$$

for all $\tau \in \{0,1\}^{m+2n}$. Note that

$$\forall \tau \in \{0,1\}^{m+2n+1} \, [\tfrac{d_{k+1}(\tau)}{d_{k+1}(\tau\restriction c_{k+1})} = 1],$$

so the new definition and the old definition of $\dot{d}$ result in the same value at level $m + 2n$. We have now incorporated another recursive martingale into $\dot{d}$.

Verification. We must consider both the possibility that Case 2 occurs infinitely often and the possibility that Case 2 occurs finitely often. If Case 2 occurs infinitely often, every martingale d_k is incorporated on some level c_k into the construction and

$$\dot{d}(R\restriction m) = 2^{-k} \tfrac{d_k(R\restriction m)}{d_k(R\restriction c_k)} + \ddot{d}(R\restriction m)$$

for some martingale $\ddot{d}$ and all $m \geq c_k$. It follows that if d_k witnesses that R is not recursively random, so does $\dot{d}$. However, by our construction, $\dot{d}$ gains at most 2^{-n} in Case 2 and loses capital in Case 1 and Case 3. Therefore, $\dot{d}$ does not demonstrate that R is not recursively random.

On the other hand, in this case, $I_{A\restriction(u(n)+1)} \subseteq R$ for infinitely many n. This means that a martingale truth-table computable from A that divides the initial capital of 1 into pieces of size 2^{-n-1} in the beginning and uses each such piece to bet $2n$ times that all the members of $I_{A\restriction(u(n)+1)}$ are in R can be constructed. If this succeeds, 2^{n-1} is gained on $I_{A\restriction(u(n)+1)}$. This strategy will succeed for infinitely many n, and this martingale will witness that R is not truth-table Schnorr random relative to A.

Now we consider the possibility that Case 2 occurs only finitely often. For almost all n, if $\sigma = A\restriction(u(n)+1)$, Case 3 occurs and the capital decreases by 8^{-n}. Furthermore, when Case 1 occurs, the capital does not increase. Therefore, the capital achieves a maximal value r at some stage. It follows that for almost all n, there are only $r \cdot 8^n$ many strings $\sigma \in J_n$ for which at least 8^{-n} is lost while following R. Furthermore, as the algorithm runs through Case 1 and Case 3 all but finitely many times, both the martingale $\dot{d}$ and the set R are recursive. This means that the $r \cdot 8^n$ strings $\sigma \in J_n$ on which the capital decreases by at least 8^{-n} can be computed and there is a recursive function g such that for almost all n, there is a number $\ell \in \{16^{n-1}+1, 16^{n-1}+2, \ldots, 16^n\}$ with $g(\ell) = \sigma$ for each such string σ. It follows that

$$A\restriction(u(n)+1) \in \{g(0), g(1), \ldots, g(16^n)\}$$

for almost all n, so $A\restriction u(n) \in \{g(0)\restriction u(n), g(1)\restriction u(n), \ldots, g(16^n)\restriction u(n)\}$. ∎

We can now develop several natural characterizations of Schnorr triviality that will be used throughout the paper.

Theorem 3.2. *The following statements are equivalent for a set A.*

1. *A is Schnorr trivial.*
2. *There is a recursive function h such that for all $f \leq_{tt} A$, there is a recursive function g such that $f(n) \in \{g(0), g(1) \ldots, g(h(n))\}$ for almost all n.*
3. *For every order function h and every $f \leq_{tt} A$, there is a recursive function g such that $\forall n\, \exists m \leq h(n)\, [f(n) = g(\langle n, m\rangle)]$.*
4. *A is truth-table low for Schnorr randomness.*

Proof. We begin by observing that saying that A is Schnorr trivial is equivalent to saying that for every recursive probability distribution μ on $\{0,1\}^*$, there is a recursive probability distribution ν on $\{0,1\}^*$ and a rational $q > 0$ such that $\nu(\{A\restriction n\}) \geq q \cdot \mu(\{0,1\}^{n+1})$ holds for all n. It is simple to replace a recursive-valued prefix-free Turing machine by a recursive probability distribution, and, given a recursive probability distribution, we can use a standard argument with Kraft-Chaitin sets to produce prefix-free Turing machines. For instance, such an argument appears in [6]. Therefore, we may use this as our characterization of Schnorr triviality.

(1.) implies **(2.)**: Assume that $f \leq_{tt} A$ is given via a tt-reduction φ_e and let $u(n)$ be the use function for φ_e. Without loss of generality, we can assume that u is not only recursive but strictly increasing. We will let $h(n) = 3^n$. Now define μ such that $\mu(\{0,1\}^{u(n)}) = 2^{-n-1}$ for all n. By (1.), there is another measure ν and a constant q such that $\nu(\{A\restriction u(n)\}) \geq q \cdot 2^{-n-1}$. Now we define a function g such that $\{g(3^{n-1}), g(3^{n-1}+1), \ldots, g(3^n-1)\}$ contains $\varphi_e^\sigma(n)$ for all σ and n such that $\nu(\{\sigma\}) > 3^{1-n}$. Note that since there are at most 3^{n-1} possible choices for σ for each n and we may have $2 \cdot 3^{n-1}$ such σ for each n in our construction, we can construct g recursively. Furthermore, $\nu(f(n)) > q \cdot 2^{-n-1}$, so $\nu(f(n)) > 3^{1-n}$ for almost all n, and g will be as required.

(2.) implies **(3.)**: Let h be the order function of (3.), and let $\tilde{h}$ be the corresponding recursive bound of (2.). Let $f \leq_{tt} A$ be given. We define the function u as $u(n) = \min\{m : h(m) > \tilde{h}(n+1)\}$ to let us translate the bound $\tilde{h}$ into h. Here, we do not consider f itself but strings of the form $f\restriction u(n)$, so any information obtained from statement (2.) actually gives us the values $f(m)$ for all $m < u(n)$. These strings can also be computed via a truth-table reduction of A, so by (2.), there is a recursive function $\tilde{g}$ such that

$$\forall^\infty n\,[f\restriction u(n) \in \{\tilde{g}(0), \tilde{g}(1), \ldots, \tilde{g}(h(n))\}].$$

This permits us to define $g(\langle m, n\rangle)$ to be the n^{th} component of $\tilde{g}(m)$ whenever $\tilde{g}(m)$ is a string of length at least n and as 0 otherwise. The function u is defined such that for every n there is a k with $\tilde{h}(k) < h(n)$ and $u(k) \geq n$. Therefore, $f\restriction u(k)$ appears among the first $h(n)$ values of $\tilde{g}$, and $f(n) \in \{g(\langle n, 0\rangle), g(\langle n, 1\rangle), \ldots, g(\langle n, h(n)\rangle)\}$.

(3.) implies **(4.)**: As an intermediate step, we show that for every martingale $d \leq_{tt} A$ and every recursive function u, there is a recursive martingale $\tilde{d}$ such that for all sets B, all $n > 0$ and all $m \leq u(n)$, the inequality $\tilde{d}(B\restriction m) \cdot n \geq d(B\restriction m)$ holds.

We assume that $d \leq_{tt} A$ and that u is a strictly increasing recursive function. We will construct a recursive martingale $\tilde{d}$ such that d and $\tilde{d}$ differ by at most a factor of n on inputs shorter than $u(n)$. To do this, let $h(n) = 2^n$ and choose f so $f(n)$ is a representation of the unique martingale d_n that satisfies

$$\forall \sigma \in \{0,1\}^{u(4^{n+1})+1}\, \forall \tau \in \{0,1\}^*\, [d_n(\sigma^\frown\tau) = d(\sigma)]$$

and has initial value 1. Now we observe that there is a function g which outputs representations of martingales such that the martingale $\tilde{d}_{\langle n,m\rangle} = g(\langle n, m\rangle)$ is recursive and such that for every n, there is an $m < 2^n$ such that $\tilde{d}_{\langle n,m\rangle} = d_n$. Now let

$$\tilde{d}(\sigma) = \sum_{n>0} \sum_{m<2^n} 4^{-n} \tilde{d}_{\langle n,m\rangle}(\sigma).$$

It is easy to see that $\tilde{d}$ is a recursive function that satisfies the equation $\tilde{d}(\sigma) = \frac{1}{2}(\tilde{d}(\sigma^\frown 0) + \tilde{d}(\sigma^\frown 1))$. Furthermore, the initial value of $\tilde{d}$ is $\sum_{n>0} \sum_{m<2^n} 4^{-n} \cdot 1 = 1$, so $\tilde{d}$ is a recursive martingale. It is also true that

$$\forall \sigma \in \{0,1\}^*\, \forall n > 0\, [|\sigma| \leq u(4^{n+1}) + 1 \Rightarrow \tilde{d}(\sigma) \geq 4^{-n} d(\sigma)]$$

since there is some $m < 2^n$ with $d(\sigma) = \tilde{d}_{\langle m,n\rangle}(\sigma)$ and the weight of $\tilde{d}_{\langle m,n\rangle}(\sigma)$ in the sum is 4^{-n}. It follows that for each $m \in \{4^n, 4^n+1, \ldots, 4^{n+1}-1\}$ and each σ of length at most $u(m)$, the inequality $\tilde{d}(\sigma) \cdot m \geq d(\sigma)$ holds.

Now we prove **(4.)**. Assume that R is a set that is not truth-table Schnorr random relative to A, so there must be a martingale $d \leq_{tt} A$ and a recursive bound function b such that

$$\exists^\infty n\, [d(R\upharpoonright b(n)) \geq n].$$

Let $u(n) = b(4n^2)$. By assumption, there is a recursive martingale $\tilde{d}$ such that

$$\forall n\, [\tilde{d}(R\upharpoonright u(n)) \cdot n \geq d(R\upharpoonright u(n))].$$

Furthermore, there are infinitely many n such that there is an $m \in \{4n^2, 4n^2+1, \ldots, 4(n+1)^2-1\}$ such that $d(R\upharpoonright b(m)) \geq m \geq 4n^2$. It follows that if such an m exists for n, the inequality $d(R\upharpoonright b(4(n+1)^2)) \geq n^2$ holds. Since $u(n+1) = b(4(n+1)^2)$, we can deduce that

$$\exists^\infty n\, [\tilde{d}(R\upharpoonright u(n)) \geq n]$$

and we are done.

(4.) implies **(2.)**: We let $h(n) = 16^n$ and apply Theorem 3.1.

(3.) implies **(1.)**: Let μ be a recursive probability distribution on $\{0,1\}^*$. Without loss of generality, $\mu(\sigma) > 0$ for all σ. There is a strictly increasing recursive function f such that

$$\forall n\, [\mu(\{\sigma : |\sigma| \geq f(n)\}) \leq 2^{-n}].$$

By (3.), there is a recursive function g such that

$$\forall n\, \forall m \leq f(n)\, [A\upharpoonright m \in \{g(\langle m, 0\rangle), g(\langle m, 1\rangle), \ldots, g(\langle m, n\rangle)\}].$$

Without loss of generality, we can assume that for every m and k, the value of $g(\langle m, k\rangle)$ is a string of length $m+1$. It follows from our choice of f that

$$\sum_{f(n) \leq k \leq f(n+1)-1} \mu(\{0,1\}^{k+1}) \leq 2^{-n}$$

and that therefore

$$\sum_{f(n) \leq k \leq f(n+1)-1} \mu(\{0,1\}^{k+1}) \cdot (n+1) \leq (n+1) \cdot 2^{-n}.$$

Now we use the fact that $\sum_n (n+1) \cdot 2^{-n} = 4$ to see that for each n and each length $k \in \{f(n), f(n)+1, \ldots, f(n+1)-1\}$, there are $n+1$ possible strings of length k which are assigned a measure of $0.1 \cdot \mu(\{0,1\})^{k+1}$. Therefore, the sum over the assigned measure belonging to n is bounded by $0.1 \cdot (n+1) \cdot 2^{-n}$ for each n and by 0.4 if summed over all n. We can distribute the remaining measure of about 0.6 to obtain a measure ν such that

$$\forall n\, [\nu(\{\sigma : f(n) \leq |\sigma| < f(n+1)\}) \geq 0.25 \cdot (n+1) \cdot 2^{-n}]$$

and

$$\forall n \forall m \leq f(n) \, [\nu(g(\langle m, n \rangle)) \geq 0.1 \cdot \mu(\{0,1\}^{m+1})].$$

It follows that

$$\forall m \, [\nu(\{A \upharpoonright m\}) \geq 0.1 \cdot \mu(\{0,1\}^{m+1})],$$

so ν has the required properties. ∎

Remark 3.3. *Since truth-table reductions depend only on the values of the oracle below the use, we can easily get an additional characterization. Let h be a recursive function such that $h(1) > h(0)$, $h(n+2) - h(n+1) \geq h(n+1) - h(n)$ for all n and $h(n+2) - h(n+1) > h(n+1) - h(n)$ for infinitely many n. A set A is Schnorr trivial if and only if for every recursive function u, there is a recursive function g such that*

$$\forall n \, [A \upharpoonright u(n) \in \{g(h(n)), g(h(n)+1), \ldots, g(h(n+1)-1)\}].$$

4 Maximal and Cohesive Sets

In this section, we produce some natural examples of Schnorr trivial sets by showing that all maximal sets are Schnorr trivial and investigate the extent to which this result can be generalized. It turns out that the proof relies heavily on the fact that maximal sets are dense simple, so some r-maximal sets are not Schnorr trivial. Furthermore, every maximal set is the complement of a cohesive set, so there are Schnorr trivial cohesive sets. However, this result does not generalize to all cohesive sets and, in fact, that only cohesive sets of high Turing degree can be Schnorr trivial.

We recall that a set A is dense simple if it is r.e. and its principal function dominates every recursive function. We further recall that A is hyperhypersimple if there is no disjoint weak array $\{F_n\}_{n \in \mathbb{N}}$ such that for all n, $F_n \cap (\mathbb{N} - A) \neq \emptyset$ and that A is maximal if A is r.e., the complement of A is infinite and there is no r.e. set W such that $W \cap A$ and $W \cap (\mathbb{N} - A)$ are both infinite.

Theorem 4.1. *Every superset of a dense simple set is Schnorr trivial. In particular, maximal and hyperhypersimple sets are Schnorr trivial.*

Proof. Let A be a dense simple set and let B be a superset of A. If u is a recursive function then for almost all n, $|\{0, 1, 2, \ldots, u(n)\} - A| \leq n$. This means that if we are given n, we can enumerate A until a stage s is found such that all but n elements below $u(n)$ are enumerated into A_s and then list the 2^n strings σ of length $u(n)+1$ such that $\sigma(x) = 1$ for all $x \in \{0, 1, 2, \ldots, u(n)\} \cap A_s$. The string $B \upharpoonright (u(n)+1)$ is among these 2^n strings, so B is Schnorr trivial by Remark 3.3. The second statement follows from the fact that every maximal set is hyperhypersimple and dense simple. ∎

We observe that we can also apply Remark 3.3 to dense simple sets to give an alternate proof of a result that appeared in [5]; namely, that there is a Π^0_1 class of Schnorr trivial sets with no recursive members. To see this, we simply produce a partial recursive function ψ that has no total extension whose domain is a maximal set and consider the class of all sets whose characteristic function extends it. Since every maximal set is dense, we can recursively bound the number of elements $< u(n)$ that are not in the domain for any n and then list a small number of possibilities for the strings of length $u(n)$ that match ψ below $u(n)$.

Now we show that Theorem 4.1 does not extend to r-maximal sets. Recall that a set A is r-maximal if A is r.e., its complement is infinite and there is no recursive set R such that $R \cap A$ and $R \cap (\mathbb{N} - A)$ are both infinite.

Theorem 4.2. *There is an r-maximal set that is not Schnorr trivial.*

Proof. Stephan [24] proved that there is an r-maximal set A that can be interpreted as a set of strings so that every string not in A is incompressible for prefix-free Kolmogorov complexity (up to a constant). Now let $f(n)$ be the lexicographically least string in $(\{0,1\}^n - A) \cup \{1^n\}$. It is clear that $f \leq_{tt} A$ and that $f(n)$ has high Kolmogorov complexity whenever $\{0,1\}^n \not\subseteq A$. The latter is true for infinitely many n, so it follows that there is no recursive function g such that

$$\forall^\infty n\, [f(n) \in \{g(0), g(1), \ldots, g(2^{n/2})\}],$$

and we can see from Theorem 3.2 (2.) that A is not Schnorr trivial. ▮

We say that A has high Turing degree if and only if the halting problem relative to K is Turing reducible to the halting problem relative to A, which is equivalent to the existence of an A-recursive function that dominates every recursive function [14]. Recall that an infinite set A is cohesive if there is no r.e. set W such that both $W \cap A$ and $W \cap (\mathbb{N} - A)$ are infinite. Jockusch and Stephan [10] showed that there are cohesive sets that do not have high Turing degree. We show now that no such set is Schnorr trivial.

Theorem 4.3. *No cohesive set of nonhigh Turing degree is Schnorr trivial.*

Proof. Let A be a cohesive set of nonhigh Turing degree and let $a_0, a_1, a_2, \ldots$ be an enumeration of A in strictly ascending order. The function $n \mapsto a_{3^n}$ is A-recursive and, as A is not high, there is a strictly increasing recursive function h such that $h(n) > a_{3^n}$ for infinitely many n. Let $I_n = \{h(n), h(n)+1, \ldots, h(n+1)-1\}$ for all n. There are infinitely many n such that $|A \cap I_n| > 2^n$, since otherwise there would be a constant c such that $h(n) \leq a_{2^{n+1}+c}$ for all n, contradicting the fact that

$\forall^\infty n\,[2^{n+1} + c < 3^n]$.

Now let $f(n) = \min((A \cap I_n) \cup \{h(n+1)\})$. This function is truth-table reducible to A. If A were Schnorr trivial, then there would be a recursive function g such that $f(n) \in \{g(2^n), g(2^n+1), \ldots, g(2^{n+1}-1)\}$ for all n. This would allow us to define the recursive set

$$B = \{g(m) : \exists n\,[2^n \leq m < 2^{n+1} \wedge h(n) \leq g(m) < h(n+1)]\}.$$

If $|A \cap I_n| > 2^n$, then the set B will contain at least one and at most 2^n of the elements of $A \cap I_n$, so there will be infinitely many n such that $A \cap I_n \cap B \neq \emptyset$ and $A \cap I_n - B \neq \emptyset$. It follows that $A \cap B$ and $A - B$ must both be infinite. This contradicts our assumption that A is cohesive, so A cannot be Schnorr trivial. ∎

This result can be generalized to show that the Turing degree of a nonhigh cohesive set does not contain a Schnorr trivial set. The basic idea of the proof is the same, but the details have to be altered somewhat.

Corollary 4.4. *If A is cohesive, $A \leq_T B$ and B is not high, then B is not Schnorr trivial.*

Proof. Assume that $A \leq_T B$ and let $u(n)$ be the use function for the computation of a_{3^n} relative to B, where $a_0, a_1, a_2, \ldots$ is a strictly ascending enumeration of A as in the previous theorem. Without loss of generality, we may suppose that u is strictly monotonically increasing and that $u(n) > a_{3^n}$ for all n. We now observe that there must be a strictly increasing recursive function h such that $h(n+1) > u(h(n))$ for infinitely many n. Otherwise, the B-recursive function $\tilde{u}$ inductively defined by $\tilde{u}(0) = 0$ and $\tilde{u}(n+1) = u(\tilde{u}(n))$ would dominate all recursive functions, which is impossible since B is not high.

Now define $f(n)$ to be the maximal element of A computed from B in $h(n+1)$ steps and assume for a contradiction that B is Schnorr trivial. Note that $f(n) \in \{g(2^n), g(2^n+1), \ldots, g(2^{n+1}-1)\}$ for all n by Remark 3.3. Now let

$$E = \{g(m) : \exists n\,[2^n \leq m < 2^{n+1} \wedge h(n) \leq g(m) < h(n+1)]\}.$$

Suppose that n is one of the infinitely many m such that $u(h(m)) < h(m+1)$. Since $h(n) \geq n$, there are more than 3^n elements of A above $h(n)$ that are computed relative to B within $h(n+1)$ steps. One of these elements is $f(n)$, which is in the set E. However, there are only 2^n elements in E between $h(n)$ and $h(n+1)$, so $A \cap E$ and $A - E$ both have an element between $h(n)$ and $h(n+1)$. It follows that $A \cap E$ and $A - E$ are both infinite, which contradicts the assumption that A is cohesive. Therefore, B cannot be Schnorr trivial. ∎

5 Reducibilities

In this section, we will consider Schnorr triviality in the context of stronger reducibilities than Turing reducibility, particularly with respect to downwards closure. However, our first result involves not only downwards closure but closure under join as well. Downey, Griffiths and LaForte [3] proved that Schnorr trivials are closed downwards under truth-table reductions and asked whether they are also closed under join. Franklin [5] gave a positive answer to this question.

Theorem 5.1 [3, 5]**.** *Let A and B be Schnorr trivial and let $C \leq_{tt} A \oplus B$. Then C is Schnorr trivial.*

Proof. Let $f \leq_{tt} C$. Then $f \leq_{tt} A \oplus B$, and we let u be the use function for this truth-table reduction. Remark 3.3 tells us that there are recursive functions g_A and g_B such that the following conditions hold for all n.

$$\begin{aligned} A\restriction(u(n)+1) &\in \{g_A(2^n), g_A(2^n+1), \ldots, g_A(2^{n+1}-1)\} \\ B\restriction(u(n)+1) &\in \{g_B(2^n), g_B(2^n+1), \ldots, g_B(2^{n+1}-1)\} \end{aligned}$$

It follows that $f(n)$ can be computed from a pair $\langle g_A(i), g_B(j)\rangle$ with $i, j < 2^n$ in such a way that the computation terminates for all such pairs, although some of these computations will probably produce incorrect values. There are 4^n such pairs and each of these pairs produces one possible value for $f(n)$, so we can construct a function g that lists exactly 4^n candidates for each n, including $f(n)$. It follows from part (2.) of Theorem 3.2 that C is Schnorr trivial. ▮

One might ask whether truth-table reducibility can be replaced by weak truth-table reducibility or bounded Turing reducibility in the theorem above. We can now show that the Schnorr trivial sets are not closed under either, so of these three, only truth-table reducibility preserves Schnorr triviality. Recall that a weak truth-table (wtt) reduction is a Turing reduction for which the use is bounded by a total recursive function and that a bounded Turing (bT) reduction is a Turing reduction for which there is a constant c that bounds the number of queries that are made to the oracle for any input. These two restrictions on Turing reducibility can be combined to generate a reducibility called bounded weak truth-table (bwtt) reducibility. We say that $A \leq_{wbtt} B$ if and only if there are a finite set of recursive functions $f_1, f_2, \ldots, f_n$ and a partial-recursive function ϑ such that $\vartheta(x, B(f_1(x)), \ldots, B(f_n(x)))\downarrow = A(x)$ for all x.

Theorem 5.2. *Given a nonrecursive r.e. set A, there is an r.e. set B such that B is not Schnorr trivial and $B \leq_{bwtt} A$. In particular, the Schnorr trivial sets are not closed under $\equiv_{bwtt}$, $\equiv_{wtt}$ and $\equiv_{bT}$.*

Proof. Let such an A be given. We begin by partitioning the natural numbers recursively into intervals $I_{\langle i,j\rangle}$ of length $2j$. For $x \in A$, we let $\Phi^A(x)$ be the stage at which x is enumerated into A; for $x \notin A$, we let $\Phi^A(x)$ be undefined.

Let $\psi(\langle i,j\rangle)$ be $\min(I_{\langle i,j\rangle} - \{\varphi_i(\langle j,k\rangle) : k < |I_{\langle i,j\rangle}| - 1\})$ whenever $j \in A$ and $\varphi_i(\langle j,k\rangle)$ is defined for all $k < |I_{\langle i,j\rangle}| - 1$ within $\Phi^A(j)$ steps. We define B to be the range of ψ. Note that B is recursively enumerable.

Now we show that $B \leq_{bwtt} A$ with the parameter $n = 1$; that is, that the reduction uses only one query and the position of the query is given by the recursive function that maps the members of each interval $I_{\langle i,j\rangle}$ to j. The reduction first determines whether j is in A. If not, $B \cap I_{\langle i,j\rangle} = \emptyset$. If so, one can compute $\Phi^A(j)$ and then check to see whether $\psi(\langle i,j\rangle)$ is defined by running each of the finitely many corresponding computations $\varphi_i(\langle j,k\rangle)$ for $\Phi^A(j)$ steps. If $\psi(\langle i,j\rangle)$ is defined, then $B \cap I_{\langle i,j\rangle}$ is equal to $\{\psi(\langle i,j\rangle)\}$. Otherwise, we will have $B \cap I_{\langle i,j\rangle} = \emptyset$. As this reduction is a bounded Turing reduction with only one query and this query is j for any $x \in I_{\langle i,j\rangle}$, the reduction is also a weak truth-table reduction. This shows that $B \leq_{bT} A$ and that $B \leq_{wtt} A$.

If B were Schnorr trivial, there would be a total recursive function φ_i such that for all j and x with $\{x\} = B \cap I_{\langle i,j\rangle}$, x would be in the set $\{\varphi_i(\langle j,k\rangle) : k < |I_{\langle i,j\rangle}| - 1\}$. It now follows from the construction of B that the computation time for the members of the set would be larger than $\Phi^A(j)$ for almost all $j \in A$, as otherwise infinitely many diagonalizations would take place. If this were the case, A would be recursive and we would have a contradiction, so B cannot be Schnorr trivial. It follows that $A \oplus B$ is not Schnorr trivial and is bwtt-, wtt- and bT-equivalent to A.

As maximal sets are Schnorr trivial, there are nonrecursive Schnorr trivial r.e. sets; further examples are provided by Franklin [5]. Therefore, the Schnorr trivial sets are not closed under $\equiv_{bwtt}$, $\equiv_{wtt}$ and $\equiv_{bT}$ and, in particular, are not closed downwards under $\leq_{bwtt}$, $\leq_{wtt}$ and $\leq_{bT}$. ▮

This result can be generalized to characterize the Schnorr trivial sets that wtt-compute only other Schnorr trivial sets.

Theorem 5.3. *Let A be Schnorr trivial. Then every $B \leq_{wtt} A$ is Schnorr trivial if and only if every function $g \leq_{wtt} A$ is majorized by a recursive function.*

Proof. Assume that there is a function $g \leq_{wtt} A$ that is not majorized by any recursive function. Without loss of generality, we can assume that g is monotonically increasing. Now we construct B so that $B \leq_{wtt} A$ and B is not Schnorr trivial. We take $B(\langle x,0\rangle) = A(x)$ for all x. If the computation of $\varphi_x(y)$ converges within $g(y)$ steps to a string such that the $\langle x, y+1\rangle^{st}$ bit of this string exists and is 0, we set $B(\langle x, y+1\rangle) = 1$. Otherwise, we define $B(\langle x, y+1\rangle) = 0$. It is easy to see that

$B \equiv_{wtt} A$.

Now we let $f(n) = B \upharpoonright \langle 2^{n+1}, 2^{n+1} \rangle$ and assume for a contradiction that φ_x is a recursive function satisfying

$$\forall n \, [f(n) \in \{\varphi_x(2^n), \varphi_x(2^n + 1), \ldots, \varphi_x(2^{n+1} - 1)\}].$$

Then there is some n such that $g(2^n)$ is greater than the maximal computation time of φ_x on inputs below $\langle 2^{n+1}, 2^{n+1} \rangle$. It follows that for $y \in \{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}$, the value of $B(\langle x, y \rangle)$ is not equal to the corresponding bit given by $\varphi_x(y)$. Hence the assumption is wrong and so, by Remark 3.3, B is not Schnorr trivial.

Now assume that every function wtt-reducible to A is majorized by a recursive function. Assume that $B \leq_{wtt} A$ and let g be the function that gives the number of steps in the computation of $B(n)$ from A for every n. Then g is majorized by a recursive function h. This lets us define a truth-table reduction from B to A which, for each n, gives the value computed if the computation terminates in $h(n)$ steps and 0 otherwise. This shows that $B \leq_{tt} A$ and therefore that B is Schnorr trivial. ∎

Miller and Martin [14] showed that for all nonrecursive $A \leq_T K$, the convergence modulus, defined as $c_A(x) = \min\{s > x : \forall y \leq x \, [A_s(y) = A(y)]\}$, is wtt-reducible to A and not majorized by any recursive function; see Odifreddi [18, Exercise XI.1.8.(b)]. This gives the following corollary when combined with the previous result.

Corollary 5.4. *If $A \leq_T K$ and A is not recursive, then some set in the wtt-degree of A is not Schnorr trivial.*

However, we can also use Theorem 4.1 to provide an example of a wtt-degree that consists entirely of Schnorr trivial sets and is contained in a high Turing degree.

Proposition 5.5. *Let A be maximal, let G be 2-generic and consider their union $A \cup G$. Every $B \leq_{wtt} A \cup G$ and every $B \leq_{bT} A \cup G$ is truth-table reducible to $A \cup G$ and thus Schnorr trivial. Furthermore, $A \cup G$ has high Turing degree.*

Proof. Let A and G be as above. By Theorem 4.1, $A \cup G$ is Schnorr trivial. Now let $B \leq_{wtt} A \cup G$. Then there must be an index e such that $B = \varphi_e^{A \cup G}$ and the use is bounded by a recursive function h. Let $G_n = G \cap \{0, 1, \ldots, n\}$ for all n.

Therefore, for any n there are an x_n and a set $E_n \subseteq \{n+1, n+2, \ldots, h(x_n)\}$ such that $\varphi_e^{G_n \cup E_n \cup A}(x_n)$ is undefined. There is a K-recursive function that computes such an x_n and E_n for every n. As G is 2-generic, there is an n such that $G \cap \{n + 1, n+2, \ldots, h(x_n)\} = E_n$. It follows that $\varphi_e^{A \cup G}(x_n)$ is undefined, which contradicts our choice of e.

Hence, by the 2-genericity of G, there must be an n such that, for all subsets $E \subseteq \{n+1, n+2, \ldots\}$ and all x, $\varphi_e^{G_n \cup E \cup A}(x)$ is defined. Then there must be some d

such that $\varphi_d^X(x)$ is defined as follows based on whether the corresponding computation or search terminates first.

Case 1: $\varphi_d^X(x) = \varphi_e^X(x)$ if the latter computation terminates.

Case 2: $\varphi_d^X(x) = 0$ if there is an $m \leq n$ such that $(G \cup A)(m) \neq X(m)$.

Case 3: $\varphi_d^X(x) = 0$ if there is an $m > n$ such that $m \in A$ and $X(m) = 0$.

Note that the second and third cases do not occur when $X = A \cup G$, so $\varphi_d^{A\cup G} = \varphi_e^{A\cup G} = B$. However, φ_d^X is total for every oracle X since the cases where it could be undefined do not occur, so $B \leq_{tt} A$. It follows that B is Schnorr trivial as well.

Now assume that $B \leq_{bT} A \cup G$. Without loss of generality, we may suppose that the computation involves exactly i queries for every input. Now let $f_1(x), f_2(x), \ldots, f_i(x)$ be the places where the queries occur. We show by induction that $f_j \leq_{wtt} A \cup G$ for all $j \leq i$. Given j, we assume that this is true for all $k < j$. Then there is a recursive upper bound on the values of all f_k with $k < j$, and we call this bound g_j. Therefore, all the queries made to calculate $f_j(x)$ are bounded by $g_j(x)$ and so $f_j \leq_{wtt} A \cup G$. This gives us a recursive function g_{i+1} such that $g_{i+1}(x) > f_k(x)$ for all $k \leq i$, so $B \leq_{wtt} A \cup G$. It follows that B is Schnorr trivial by the previous paragraph of this proof.

Note that $A \cup G$ is coinfinite since G is 2-generic. Furthermore, as A is dense simple and $A \cup G \supseteq A$, the complement of $A \cup G$ is also dense immune. Therefore, the function mapping n to the n^{th} nonelement of $A \cup G$ dominates every recursive function and it follows that $A \cup G$ has high Turing degree. ∎

The wtt-degree of $A \cup G$ contains only Schnorr trivial sets. Since the class of 2-generic sets is closed under complementation, the wtt-degree of $A \cup (\mathbb{N} - G)$ also consists only of Schnorr trivial sets. However, as $A = (A \cap G) \cap (A \cup (\mathbb{N} - G))$, A is wtt-reducible to the join of the two sets and there is a set $B \equiv_{wtt} A$ which is not Schnorr trivial. Therefore, the class of those sets whose wtt-degree entirely consists of Schnorr trivial sets is not closed under join.

In the revision of [23], Soare emphasizes the distinction between every function in a degree being majorized by a recursive function and a degree not containing a hyperimmune set. While these notions are the same for Turing degrees, this example shows that they differ for wtt-degrees; that is, $A \cup G$ satisfies the former but fails to satisfy the latter, as $\mathbb{N} - (A \cup G)$ is hyperimmune. It is not too difficult to construct a set of hyperimmune Turing degrees such that its wtt-degree has both of these properties. For instance, we may consider $R \oplus G$, where R is a hyperimmune-free Martin-Löf random set and G is 2-generic relative to R. We may ask whether such a construction can be combined with the notion of Schnorr triviality.

Question 5.6. *Is there a Schnorr trivial set A of hyperimmune Turing degree such that every function weakly truth-table reducible to A is majorized by a recursive function and the weak truth-table degree of A does not contain a hyperimmune set?*

Finally, we present a sort of downwards density theorem. Downey, Griffiths and LaForte [3] showed that there is an r.e. Turing degree which does not contain a Schnorr trivial set. The next proof shows that, nevertheless, every nonrecursive r.e. set bounds a nonrecursive Schnorr trivial set.

Theorem 5.7. *Let A be r.e. and nonrecursive. Then there is an r.e. nonrecursive set $B \leq_{bwtt} A$ that is Schnorr trivial and nonrecursive.*

Proof. Since A is not recursive, there is a high r.e. set C such that $A \not\leq_T C$ [20]. Without loss of generality, we can choose C to be dense simple and co-retraceable via a total function, so there is a recursive function h such that $h(x) = |\{y < x : y \notin C\}|$ for all $x \notin C$. Let $a_0, a_1, a_2, \ldots$ be a recursive injective enumeration of A and let $c_0, c_1, c_2, \ldots$ be a recursive injective enumeration of C. Now let

$$b_n = \min\{x : h(x) = a_n \wedge x \notin \{c_0, c_1, \ldots, c_n\}\}.$$

Note that each b_n is defined, since there is an $x \notin C$ with $h(x) = a_n$ for every a_n. The set $B = \{b_0, b_1, b_2, \ldots\}$ is recursively enumerable. Furthermore, we can see that $B \leq_{bwtt} A$ as follows. Given any x, if $h(x) \notin A$, then $x \notin B$. If $h(x) \in A$, then let n be the unique number such that $x = a_n$, so we will have $x \in B$ if and only if $x = b_n$ for this n.

The restriction of the characteristic function of B to C is a partial recursive function ψ. Given any $x \in C$, we can find the first n such that $x = c_n$ and then let $\psi(x) = 1$ if and only if there is an $m < n$ such that $b_m = x$. As C is dense simple, it follows from Remark 3.3 that B is Schnorr trivial.

Furthermore, we can see that $A \leq_T B \oplus C$ as follows. If we use C as an oracle, then given any x, we can find the least y such that $h(y) = x$ and $h(y) \notin C$. Therefore, $x \in A$ if and only if there is a $z \leq y$ such that $h(z) = x$ and $z \in B$. Since $A \not\leq_T C$, we can see that $B \not\leq_T C$, so B cannot be recursive. ▮

Franklin [8] showed that if A is a nonhigh 1-generic set, there is no nonrecursive Schnorr trivial set $B \leq_T A$. As there are 1-generic sets below K and these sets are not high, we can see that Theorem 5.7 cannot be improved to show that every nonrecursive $A \leq_T K$ bounds a nonrecursive Schnorr trivial set.

6 Reductions to Schnorr Random Sets

It is natural to ask whether there is a parallel to Hirschfeldt and Nies's characterization of the sets that are low for Martin-Löf randomness as those sets that are bases for

Martin-Löf randomness. While we show that there is no such equivalence for Schnorr randomness, there is a promising initial proposition.

Proposition 6.1. *Let A and B be sets such that $A \leq_{tt} B$ and B is truth-table Schnorr random relative to A. Then A is Schnorr trivial.*

Proof. Let A and B be as above and let $h(n) = 4^n$. Define u to be the use function of the truth-table reduction from A to B and $r(n)$ to be the number of strings in $\{0,1\}^{u(n)+1}$ that truth-table compute $A\restriction n$ via the given reduction. Furthermore, let f be an arbitrary recursive, strictly increasing function.

For each n, we define the martingale d_n to have the initial value 2^{-n}, increase on any string of length $u(f(n))+1$ computing $A \restriction f(n)$ via the given truth-table reduction up to the value $2^{u(f(n))+1-n}/r(f(n))$ and have the value 0 otherwise. For all $\sigma \in \{0,1\}^{u(f(n))+1}$ and all $\tau \in \{0,1\}^*$, let $d_n(\sigma^\frown\tau) = d_n(\sigma)$. This leaves two possibilities.

First, we consider the possibility that $d_n(B\restriction u(f(n))) > 2^n$ for infinitely many n. If this were to happen, then B could not be Schnorr random relative to A because the sum $\sum_n d_n$ would be an A-recursive martingale that succeeds on B. Since B is truth-table Schnorr random relative to A, this would give us a contradiction.

Therefore, $d_n(B\restriction u(f(n))) \leq 2^n$ for almost all n. It follows that $r(n) \geq 2^{u(n)+1-4n}$ for almost all n. Given n, we can produce the list of strings $\sigma \in \{0,1\}^{f(n)+1}$ for which there are at least $2^{u(n)+1-4n}$ strings $\tau \in \{0,1\}^{u(f(n))+1}$ such that the truth-table reduction from A to B translates τ into σ. There will be no more than 4^n such strings, and for almost all n, this list will contain $A\restriction f(n)$. Therefore, by Remark 3.3, A must be Schnorr trivial. ∎

However, we can generalize Calude and Nies's result [2] that the halting problem is not truth-table reducible to any Martin-Löf random set to the context of a dense simple (and thus Schnorr trivial) set and Schnorr randomness.

Theorem 6.2. *There is a partial recursive $\{0,1\}$-valued function ψ whose domain is dense simple such that no set whose characteristic function extends ψ is truth-table reducible to any Schnorr random set.*

Proof. In this proof, we will define intervals I_m of length $2m+1$ that might move from time to time after they are initially defined. Each time an interval moves, the characteristic function ψ is defined on the previous values of I_m and no other interval will ever contain these values again. We use a priority construction with a bookkeeping set R of pairs $\langle e, m\rangle$ for requirements that have already been satisfied. At the beginning, all intervals are undefined and R is initialized as the empty set.

Construction. At stage s, we first determine the least m such that I_m requires attention as defined below.

1. I_m requires attention if the interval $I_{m,s}$ is undefined at the current stage.
2. I_m requires attention with respect to $\varphi_e(x)$ if $I_{m,s}$ is defined, $\varphi_{e,s}(x)$ is defined, $e, x \leq (m+1)^2$ and $\min(I_{m,s}) \leq \varphi_e(x)$.
3. I_m requires attention with respect to the e^{th} candidate for a *tt*-reduction if $I_{m,s}$ exists, this candidate is defined for all elements of $I_{m,s}$ within s steps and $\langle e, m\rangle$ is not yet in R.

Note that there is always some interval I_m that requires attention at stage s. Let m be the minimal index of such an interval and let y be the first number which is not in a defined interval $I_{k,s}$ or in the domain of ψ_s.

If I_m requires attention because $I_{m,s}$ is undefined, then we define it by letting $I_{m,s+1} = \{y, y+1, y+2, \ldots, y+2m\}$. If I_m requires attention with respect to $\varphi_e(x)$, then we define $\psi_{s+1}(x) = 0$ for all $x \in I_m$ and let $I_{m,s+1} = \{y, y+1, y+2, \ldots, y+2m\}$. Finally, if I_m requires attention with respect to the e^{th} candidate for a truth-table reduction, then we consider all possible $\sigma \in \{0,1\}^{2m+1}$ and select the one for which the e^{th} candidate has the smallest quantity of inverse images producing it. In this case, we define ψ_{s+1} according to σ on $I_{m,s}$, let $I_{m,s+1} = \{y, y+1, y+2, \ldots, y+2m\}$ and put $\langle e, m\rangle$ into R.

Verification. It is easy to see that every interval I_m requires attention only finitely often. Therefore, each I_m has a final value $I_{m,\infty}$ and the domain of ψ is exactly the complement of the union of all intervals $I_{m,\infty}$. It is easy to see that ψ is partial recursive.

Assume now that the e^{th} candidate is a truth-table reduction. Each I_m will receive attention with respect to the e^{th} candidate at some stage and we let $u(m)$ be its use. The values of ψ on $I_{m,s}$ are such that at most $2^{u(m)-2m}$ out of the $2^{u(m)+1}$ strings of length $u(m)+1$ are mapped by the e^{th} candidate to a string that extends ψ on $I_{m,s}$, and we can find these strings effectively. Therefore, a martingale $\sum_m d_m$ can be constructed such that for each m, d_m has 2^{-m-1} as its initial value and reaches the value 2^m after querying $u(m)+1$ bits if the bits queried produce a string consistent with ψ on $I_{m,s}$. The term d_m in the sum of the martingale will be constant after querying the first $u(m)+1$ bits. It is easy to see that $\sum_m d_m$ witnesses the statement that no extension of ψ is truth-table reducible to a Schnorr random set via the e^{th} candidate for a truth-table reduction.

Now we show that the domain of ψ is dense simple. Note that the complement of the domain of ψ is given by the union $\cup_m I_{m,\infty}$. Let e be an index for which φ_e is total and let $x \geq e^2$. Any interval I_m with $(m+1)^2 \geq x$ requires attention with respect to $\varphi_e(x)$ whenever $\min(I_{m,s}) \leq \varphi_{e,s}(x)$ and the right-hand side is defined, so $\min(I_{m,\infty}) > \varphi_e(x)$. Only the intervals $I_{m,\infty}$ with $(m+1)^2 < x$ can contain elements below $\varphi_e(x)$ and these intervals have at most x elements combined. For example, if

$e = 2$ and $x = (2+1)^2 + 1$ then only the intervals $I_{0,\infty}$, $I_{1,\infty}$ and $I_{2,\infty}$ can contain elements below $\varphi_e(x)$ and these intervals have at most 1, 3 and 5 elements, respectively. Therefore, there are at most 9 elements of the complement of the domain of ψ below $\varphi_e(x)$ for $x = 10$, and we can see that the domain of ψ must be dense simple. ∎

In fact, every set whose characteristic function extends the function ψ constructed in Theorem 6.2 is Schnorr trivial since the domain of ψ is dense simple. Furthermore, ψ has a total extension A that is hyperimmune-free. Therefore, truth-table Schnorr randomness relative to A coincides with Turing Schnorr randomness relative to A, and the statements in Proposition 2.1 coincide. This indicates that the negative result of Theorem 6.2 does not depend on the particular version of truth-table Schnorr randomness relative to A and shows that it is not possible to extend the characterization to what might be called a truth-table basis for Schnorr randomness.

Now we present a counterpart to Theorem 6.2.

Theorem 6.3. *There is a dense simple set A that is Schnorr trivial and truth-table reducible to a Schnorr random set.*

Proof. Without loss of generality, assume that $\varphi_0(n) = 0$ for all n. Now we can define a recursive injective function f such that the range of f is $\{\langle e, n\rangle : e < n \wedge \varphi_e(n)\downarrow\}$ and if $f(x) = \langle e, n\rangle$, then $\varphi_e(n) \leq x$.

We will use the fact that φ_0 is always 0 to define $f(x) = \langle 0, n\rangle$ for n whenever no other value can be found. Now let $g(x)$ be the second component of $f(x)$; that is, if $f(x) = \langle e, n\rangle$, then $g(x) = n$. Let

$$A = \{x : \exists y > x\, [g(y) = g(x)]\}.$$

The set A is clearly recursively enumerable. Furthermore, A is dense simple, since for every total function φ_e and every $n > e$, if $f(x) = n$ and $x \notin A$, then $x \geq \varphi_e(n)$.

Now let $I_0, I_1, I_2, \ldots$ be a recursive partition of $\mathbb{N}$ into intervals such that $|I_m| = g(m)$ for all m, and let d be a universal martingale that succeeds on all sets that are not Martin-Löf random. Note that d is approximable from below but not recursive, since its initial value is a left-r.e. real between 0 and 1.

We define a set R inductively as follows. On an interval I_m with $m \in A$, we choose R such that d grows by at most a factor $1/(1-2^{-m})$ and is not 0 on all elements of I_m. No element of an interval I_m with $m \notin A$ is put into R, and d can grow by a factor of 2^m on this interval. It is clear that $A \leq_{tt} R$, since $m \in A$ if and only if $I_m \cap R \neq \emptyset$.

Assume now for a contradiction that R is not Schnorr random. Let r be the factor by which d can grow on intervals I_m with $m \in A$; that is,

$$r = \prod_{m>0} (1-2^{-m})^{-m}.$$

Note that this product is convergent. As R is assumed not to be Schnorr random, there is a recursive function h such that $d(R\upharpoonright h(n)) > n$ for infinitely many n. Choose an e such that $\varphi_e(m) = h(0) + h(1) + \ldots + h(\lceil r \cdot 2^{(m+1)(m+2)} \rceil) + m$ for all m. By construction there are, for $m > e$, only m intervals I_n such that $I_n \cap R = \emptyset$ below $\varphi_e(m)$ and these intervals have lengths $1, 2, 3, \ldots, m$, respectively. On these m intervals, d may increase its value by a factor of $2^{m(m+1)}$. Outside the intervals I_n such that $I_n \cap R = \emptyset$, the value of d grows by at most a factor of r. As a consequence, for all $m > e$ and all k with $r \cdot 2^{m(m+1)} < k \leq r \cdot 2^{(m+1)(m+2)}$,

$$d(R\upharpoonright h(k)) \leq r \cdot 2^{m(m+1)} < k.$$

Therefore, the assumption that d reaches the value k after the first $h(k) + 1$ bits for infinitely many k is false, so R is Schnorr random and A is truth-table reducible to a Schnorr random set. ▮

This leads us to the following question.

Question 6.4. *Do the Schnorr trivial sets that are truth-table reducible to Schnorr random sets form a truth-table ideal? In other words, if A and B are both Schnorr trivial sets that are truth-table reducible to a Schnorr random set, is the same true for $A \oplus B$?*

One might try to prove a similar statement by considering alternative reducibilities; in particular, those for which the Schnorr trivial sets form the least degree. For instance, we could consider the following reducibility as well as the $\leq_{Sch}$ considered by Downey, Griffiths and LaForte [3].

Definition 6.5. We say that $A \leq_{snr} B$ if and only if

$$\exists \text{ recursive } h \, \forall f \leq_{tt} A \, \exists g \leq_{tt} B \, \forall n \, \exists m \leq h(n) \, [f(n) = g(m)].$$

We can see by Theorem 3.2 (2.) that $A \leq_{snr} \emptyset$ if and only if A is Schnorr trivial. This result can now be extended to obtain a theorem similar to those involving bases for randomness, although the reducibility is not a commonly accepted one. Note that for every fixed set B, the class $\{A : A \leq_{snr} B\}$ is a truth-table ideal.

Theorem 6.6. *A set A is Schnorr trivial if and only if there is a set B such that B is truth-table Schnorr random relative to A and $A \leq_{snr} B$.*

Proof. If A is Schnorr trivial, then A is snr-reducible to every set, so we need only prove the other direction. Assume that $A \leq_{snr} B$ and that B is truth-table Schnorr random relative to A. We will show that A is Schnorr trivial by arguments adapted from the proof of Proposition 6.1.

Let the recursive bound h from the snr-reduction be given and let f be a strictly increasing recursive function. We will list $4^n \cdot h(n)$ strings that will include $A\restriction f(n)$ for every n.

Let u be the use function for the truth-table reduction that computes up to $h(n)$ strings, including $A\restriction f(n)$, for any given n, and let $r(n)$ be the number of strings of length $u(n)+1$ such that $A\restriction f(n)$ is among the $h(n)$ candidates produced by the given truth-table reduction.

For each n, consider the martingale d_n that has 2^{-n} as its initial value and reaches the value $2^{u(n)+1-n}/r(n)$ on any string $\sigma \in \{0,1\}^{u(n)+1}$ that produces $h(n)$ candidates, including $A\restriction f(n)$, via the given truth-table reduction. For all $\sigma \in \{0,1\}^{u(n)+1}$, if $\tau \supset \sigma$, we let $d_n(\tau) = d_n(\sigma)$.

First, we consider the possibility that $d_n(B\restriction u(n)) > 2^n$ for infinitely many n. Then B cannot be Schnorr random relative to A, since the sum $\sum_n d_n$ is an A-recursive martingale that succeeds on B. As B is Schnorr random relative to A, this cannot occur.

Therefore, $d_n(B\restriction u(n)) \leq 2^n$ for almost all n. It follows that $r(n) \geq 2^{u(n)+1-4n}$ for almost all n. For each n, we produce the list of all $\sigma \in \{0,1\}^{f(n)+1}$ such that there are at least $2^{u(n)+1-4n}$ strings $\tau \in \{0,1\}^{u(n)+1}$ for which σ is among the $h(n)$ candidates which the truth-table reduction produces for $A\restriction f(n)$ from τ. The list of such σ contains at most $4^n \cdot h(n)$ members and, for almost all n, the correct string $A\restriction f(n)$ will be found in the list. By Remark 3.3, A must be Schnorr trivial. ∎

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