LOWNESS AND HIGHNESS PROPERTIES FOR RANDOMNESS NOTIONS

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Abstract. Given two relativizable classes R and P and a real A, we say that A is in Low(R,P) if $\mathsf{R} \subseteq \mathsf{P}^A$ and that A is in High(R,P) if $\mathsf{R}^A \subseteq \mathsf{P}$. In this paper, we survey the current results on highness and lowness for Kurtz, Schnorr, recursive, Martin-Löf, and weak 2-randomness.

Contents

1. Introduction

Lowness and highness are notions of relative computational strength. A real is said to be low for a relativizable class if the class generated using this real as an oracle is no different from the class generated using no oracle at all, and a real is said to be high for a relativizable class if using it as an oracle is equivalent to using the strongest possible oracle for some particular function. In this paper, we will discuss lowness and highness notions for algorithmic randomness and provide an overview of the existing results.

The most familiar and perhaps oldest example of these notions is based on the Turing jump: a real A is said to be low if $A' \equiv_T 0'$ and high if $A' \equiv_T 0''$. These terms were first published in a paper by Soare in 1972 [31], but applied only to the Δ^0_2 degrees with these properties. Later, the concept of lowness was developed in other contexts, such as lowness for EX-learning [30]. We formalize a more general notion of lowness below.

Definition 1.1. Let R be a relativizable class. We say that a real A is *low for* R if $\mathsf{R} = \mathsf{R}^A$. The class of reals that is low for R will be denoted here by Low(R).

We note that while lowness can clearly be defined for any relativizable class, it is more difficult to define highness in any sort of generality. The definition of A as a high Turing degree does not

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involve a class relativized to A, but rather the equality of the jump operator relativized to A and the maximal element of the Δ^0_2 reals, $0'$. However, in other cases, there may not be a natural choice for such a maximal oracle and function.

These definitions were extended to lowness and highness for pairs of related classes instead of simply single classes as below in [16] and [11], respectively.

Definition 1.2. Suppose that R and P are relativizable classes. Given a real A, we say that A is in Low(R,P) if $\mathsf{R} \subseteq \mathsf{P}^A$ and that A is in High(R,P) if $\mathsf{R}^A \subseteq \mathsf{P}$.

In short, P^A will always be a subclass of P. A real is in Low(R,P) if this subclass still contains R. On the other hand, we say that a real is in High(R,P) if its use as an oracle with respect to R generates a class of reals contained entirely within P. Note that for any notion R, Low(R,R) is precisely the class of reals that are low for R and that for any notions $\mathsf{R} \subseteq \bar{\mathsf{R}} \subseteq \bar{\mathsf{P}} \subseteq \mathsf{P}$, Low($\bar{\mathsf{R}}$,$\bar{\mathsf{P}}$) $\subseteq$ Low(R,P) and High(R,P) $\subseteq$ High($\bar{\mathsf{R}}$,$\bar{\mathsf{P}}$).

Although we have defined Low(R,P) and High(R,P) in full generality above, we will restrict our attention to Low(R,P) for R and P such that $\mathsf{R} \subseteq \mathsf{P}$ and High(R,P) for R and P such that $\mathsf{R} \supset \mathsf{P}$. The reason for this is that if $\mathsf{P} \subset \mathsf{R}$, we will have Low($\mathsf{R}$,$\mathsf{P}$) $= \emptyset$ and that if $\mathsf{R} \subseteq \mathsf{P}$, we will have High($\mathsf{R}$,$\mathsf{P}$) $= 2^\omega$.

We note that Low(P) can often be computed easily from Low(P,R) for some other notion R.

In Sections 3 through 6, we present results concerning lowness for pairs of randomness notions, and in Section 7, we present results concerning highness for pairs of randomness notions. Finally, in Section 8, we summarize and comment on these results and present some currently open questions. In general, we will not give detailed, technical proofs, but rather thorough outlines and references to the appropriate sources for the full proofs.

1.1. **Definitions and notation.** Our notation is standard and generally follows Soare [32] and Odifreddi [26, 27]. We refer to the elements of the Cantor space, 2^ω, as reals, and μ will always denote the standard Lebesgue measure on this space. We write $\tau \preceq \sigma$ to indicate that τ is an initial segment of σ, $\tau\sigma$ to denote the concatenation of τ and σ, and $[\sigma]$ to denote the set of reals extending σ. If S is a subset of $2^{<\omega}$, we define $[S]$ similarly. Finally, for any string σ and any measurable set R, we will use $\mu_\sigma(R)$ to represent the fraction of R that extends σ: $2^{|\sigma|}\mu(R \cap [\sigma])$.

For a general reference on randomness, please see Downey and Hirschfeldt [7], Nies [24], or Downey, Hirschfeldt, Nies, and Terwijn [8]; however, we will remind the reader of the most important definitions below. There are three primary approaches one can take when formalizing notions of randomness, and each of these has advantages and disadvantages when it comes to particular randomness notions. In this paper, we will only present the approaches that are most convenient for the theorems we are interested in. Any of the references above will provide equivalent characterizations using the other approaches for a given randomness notion.

We begin by recalling the definition of the most commonly studied randomness notion, Martin-Löf randomness. All three characterizations will be used at some point in this paper: that of prefix-free Kolmogorov complexity relative to a universal Turing machine, the unpredictability definition, and the test definition. If $M : 2^{<\omega} \to 2^{<\omega}$ is a prefix-free Turing machine, the complexity of a string σ with respect to M is $K_M(\sigma) = \min\{\tau \mid M(\tau) = \sigma\}$. When a universal Turing machine is used, the subscript will be omitted. For more background on Kolmogorov complexity, please see [19]. Also recall that a Martin-Löf test is a uniformly r.e. sequence $\langle V_i \rangle_{i\in\omega}$ of Σ^0_1 classes such that $\mu(V_i) \leq 2^{-i}$ for each i.

A martingale is a function $d : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ that satisfies the equation

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}$$

for all $\sigma \in 2^{<\omega}$. We may think of a martingale d as a function that tells us, for each string σ, how much capital we would have if we started with $d(\langle\rangle)$ and bet on the bits of σ in order using an associated betting strategy. An r.e. (recursive) martingale is simply a martingale whose values are uniformly r.e. (recursive) reals. We say that a martingale d *succeeds* on a real A if $\limsup_n d(A{\upharpoonright}n) = \infty$; i.e., if there is no bound on the amount of capital d can have after betting on some initial segment of A.

Definition 1.3. [20, 28] A real A is Martin-Löf random if the following three equivalent conditions hold. We will denote the class of such reals by ML.

1. $(\exists c)(\forall n)[K(A{\upharpoonright}n) \geq n - c]$.
2. No r.e. martingale d succeeds on A.
3. For every Martin-Löf test $\langle V_i \rangle_{i\in\omega}$, $A \notin \cap_i V_i$.

Note that there is a universal Martin-Löf test; i.e., a Martin-Löf test $\langle U_i \rangle_{i\in\omega}$ such that for all Martin-Löf tests $\langle V_i \rangle_{i\in\omega}$, $\cap_i V_i \subseteq \cap_i U_i$ [20]. There is also a universal r.e. martingale.

To define recursive randomness, we will only use the martingale notion.

Definition 1.4. [29] A real is recursively random if no recursive martingale succeeds on it. We will denote this class of reals by Rec.

When we discuss Schnorr randomness together with Martin-Löf randomness, the most natural characterization is in terms of tests, but when we discuss it together with recursive randomness, the most natural characterization is the martingale characterization. We present both below.

Definition 1.5. [29] The following two statements are equivalent and characterize Schnorr randomness for a real A. We will denote the class of Schnorr random reals by $\mathsf{Schnorr}$.

1. If $\langle V_i \rangle_{i\in\omega}$ is a Martin-Löf test such that $\mu(V_i) = 2^{-i}$ for each i, then $A \notin \cap_i V_i$.
2. There is no recursive martingale d such that $d(A \upharpoonright n) \geq h(n)$ infinitely often for some unbounded, nondecreasing, recursive function h.

We will refer to an unbounded, nondecreasing, recursive function as an order function from this point onwards.

The three notions presented above are the most commonly considered randomness notions we will discuss in this paper. However, we will also consider two other notions: weak 2-randomness, which is stronger than all of these, and Kurtz randomness, which is weaker. Both were introduced in [18]. Kurtz randomness was originally called weak 1-randomness, and it should be noted that weak 2-randomness is sometimes called Kurtz 2-randomness [7, 8] or strong randomness [11].

Kurtz's original definition of weak 2-randomness is listed first in the definition below.

Definition 1.6. [18, 13, 35] The following two statements are equivalent to weak 2-randomness for a real A. We will denote the class of weakly 2-random reals by $\mathsf{W2R}$.

1. For every Σ^0_2 set U of measure 1, $A \in U$.
2. $A \notin \cap_i U_i$ for any recursive sequence of r.e. open sets $\langle U_i \rangle_{i\in\omega}$ such that $U_i \supseteq U_{i+1}$ for all i and $\lim_i \mu(U_i) = 0$. Such a sequence is called a generalized Martin-Löf test.

Finally, we present Kurtz randomness.

Definition 1.7. [18] A real A is Kurtz random if for every r.e. open set U of measure 1, $A \in U$. Such a set U is called a (positive) Kurtz test. We will denote this class of reals by Kurtz.

Finally, we describe the relations between these classes of reals.

Theorem 1.8. [29, 35, 18] *The following chain of inclusions holds, and none of them are reversible.*

$$\mathsf{W2R} \subset \mathsf{ML} \subset \mathsf{Rec} \subset \mathsf{Schnorr} \subset \mathsf{Kurtz}$$

We conclude this section with the Kraft-Chaitin Theorem, which will be used frequently throughout the paper.

Theorem 1.9 (Kraft-Chaitin Theorem [4])**.** *Let $\langle d_i, \sigma_i \rangle_{i \in \omega}$ be a recursive sequence with $d_i \in \omega$ and $\sigma_i \in 2^{<\omega}$ for all i such that $\sum_i \frac{1}{2^{d_i}} \leq 1$. (Such a sequence is called a Kraft-Chaitin set, and each element of the sequence is called a Kraft-Chaitin axiom.) Then there are strings τ_i and a prefix-free machine M such that dom$(M) = \{\tau_i \mid i \in \omega\}$ and for all i and j in ω,*

(1) *if $i \neq j$, then $\tau_i \neq \tau_j$,*
(2) *$|\tau_i| = d_i$,*
(3) *and $M(\tau_i) = \sigma_i$.*

The Kraft-Chaitin Theorem allows us to construct a prefix-free Turing machine by specifying only the lengths of the strings in the domain rather than the strings themselves. We will therefore occasionally identify $\langle \tau, \sigma \rangle$ with $\langle d, \sigma \rangle$, where $d = |\tau|$.

2. Lowness for Martin-Löf randomness

The class Low(ML) is a particularly interesting one. It is unusual in that its only known characterizations involve other randomness notions (and, as we shall see, every other class we discuss with this property is identical to Low(ML)). We present some of these equivalent notions here. We begin by defining two other ways in which a real can be said to be "far from Martin-Löf random."

Definition 2.1. A real A is *K-trivial* if there is some c such that for all n, $K(A \upharpoonright n) \leq K(0^n) + c$, and A is *low for K* if there is some c such that for all $\sigma \in 2^{<\omega}$, $K(\sigma) \leq K^A(\sigma) + c$.

Informally, A is K-trivial if every initial segment of A can be described in a way that is no more complicated than a description of a string of 0s of the same length (up to a constant), and A is low for K if using A as an oracle does not materially alter the complexity of any string σ. Since Martin-Löf randomness can be defined in terms of prefix-free complexity as in Definition 1.3, every real that is low for K is also low for Martin-Löf randomness and vice versa.

Here, we show that the K-trivial reals are precisely those that are low for K, so as a corollary, we can see that the K-trivial reals are precisely those that are low for Martin-Löf randomness. Theorem 2.3 was proven by Nies and Hirschfeldt and first appeared in [23].

Proposition 2.2. *Every real that is low for K (and thus every real that is low for Martin-Löf randomness) is K-trivial.*

Proof. Suppose that c_0 witnesses A's lowness for K. Given a universal prefix-free machine U, we can find another universal prefix-free machine M such that $M^X(\sigma) = X \upharpoonright |U(\sigma)|$ for every X and

σ whenever either of these terms is defined. Then there is a constant c_1 such that for all reals X, $K^X(X\restriction n) \leq K(0^n) + c_1$ for every n, and we can see that

$$K(A\restriction n) \leq K^A(A\restriction n) + c_0 \leq K(0^n) + c_0 + c_1$$

for every n. Therefore, A is K-trivial. □

Theorem 2.3. [23] *Every K-trivial real is low for K.*

Proof. This proof utilizes Nies's "golden run" method. Due to the complicated nature of this construction, we will provide very few technical details and simply give a general outline of the proof and a description of the main ideas involved.

Let A be K-trivial, and let b witness this fact. We begin by noting that A is Δ^0_2: A is a path on the tree $T = \{\sigma \mid (\forall \rho \subseteq \sigma)[K(\rho) \leq K(0^{|\rho|}) + b]\}$, and all paths on this tree are isolated [4]. Therefore, we can choose a recursive approximation $\langle A_s \rangle_{s\in\omega}$ of A.

Two Kraft-Chaitin sets will appear in this proof. The first, L, will allow us to make use of the fact that A is K-trivial. The second, W, will witness A's lowness for K.

To construct the set W, we use a tree of runs of procedures. At each branching node, we will try to ensure that W is a witness to A's lowness for K. However, problems may arise if our approximation to A changes. The automatic inclination would be to add more elements to L to compensate for the change; however, we would risk increasing the measure of each set to the extent that neither would be a Kraft-Chaitin set. Therefore, we must develop another system to handle these changes. We will say that each axiom $\langle r, n\rangle$ that enters L must either reach an acceptable level of the tree or be garbage. The measure of each of these categories will be bounded above by 2^{-1}, which will guarantee that the measure requirement, at least, of the definition of a Kraft-Chaitin set is met for L.

We may assume that a machine M_d and its index d are given, and we will build L to be a Kraft-Chaitin set for M_d. For the rest of the proof, we will let $c = b + d$ and $k = 2^{c+1}$. If $\langle r, n\rangle$ enters L, we will ensure that $K(A\restriction n) \leq K(0^n) + b \leq r + d$.

The first technical definition we need is that of a j-set for $1 \leq j \leq k$. We say that a finite subset E of ω is a j-set at a stage t if for all $n \in E$, some axiom $\langle r_n, n\rangle$ entered L at a stage $u < t$ and there are j different strings of the form $A_v\restriction n$ at stages $u \leq v \leq t$ such that $K_v(A_v\restriction n) \leq r_n + c$. An r.e. set $E = \cup_t E_t$ is a j-set if for every t, E_t is a j-set at stage t. We note here that if E is a k-set, we can bound its measure as follows. If we put a description of 0^n into L, any matching description that might also go in must be within c of n, so for any k-set E, $\mu(E) \leq 2^{-1}$.

We now describe our tree of runs of procedures. At each stage, our tree will have $2k-2$ levels. The root procedure will be P_k, which will call several different procedures of type Q_{k-1}. Each of these will call the procedure P_{k-1}, and each of those will call several different procedures of type Q_{k-1}, and so on until we reach the procedures of type Q_1. Each of these procedures has an associated goal, which is some given amount of measure. The runs of each of these procedures will enumerate a set, and the enumeration will only stop if it reaches its goal or if the run is cancelled by runs of procedures that appear above it on the tree. The procedure P_k has 1 as its goal, and the set we will try to enumerate is a Kraft-Chaitin set of measure 1 that we will call C_k. If C_k never achieves a measure of 1, we will see that there must be a run of some procedure P_i that never returns, even though all of its subprocedures will either return or be cancelled. This "golden run" will enable us to define our set W.

In order to reach C_k, a number must get through j-sets C_j and D_j for all $1 \leq j < k$ first in the order $C_1, D_1, \ldots, D_{k-1}, C_k$. A procedure P_i will move a number n from D_{i-1} to C_i when the approximation $A \upharpoonright n$ changes. This has the effect of adding a new string of the form $A_u \upharpoonright n$ to an $(i-1)$-set and creating an i-set, guaranteeing us that C_i will be an i-set for any i. The procedure Q_1 will enumerate C_1, and the procedures of type Q_i for larger i will move numbers from C_i to D_i.

A procedure of type Q_i will be indexed by a 4-tuple $\langle i, \sigma, \tau, w \rangle$, where σ is a description, τ is the object being described, and w is a particular A-use. The procedure P_i will call procedures $Q_{i,\sigma,\tau,w}$ such that $U^A(\sigma) = \tau$, where U is a particular universal prefix-free Turing machine.

The goal of a procedure P_i is the amount of measure it wants to move from D_{i-1} to C_i, and the goal of a procedure of type Q_i is the amount of measure it wants to move from C_i to D_i. If a procedure of type Q_i returns a set D, P_i waits for a change in $A \upharpoonright w$ for the appropriate w. If one occurs, it will add D to C_i. If $A \upharpoonright w$ changes before the appropriate Q_i returns, that change will be enough to convert Q_i's current set D to an i-set, so P_i can add D to C_i immediately and cancel the run of Q_i. The additions of these weights are accompanied by the additions of approximations to initial segments of A, so we will be able to keep track of the behavior of U relativized to A.

It is the weights of the cancelled runs and the weights of the procedures that return but for which $A \upharpoonright w$ never changes that make up the "garbage" component of the set L. To control the size of this garbage component, we associate a garbage quota with each run of a procedure. For a procedure P_i, this is the amount it is allowed to leave in $D_{i-1} - C_i$; for a procedure of type Q_i, this is the amount it is allowed to leave in $C_i - D_i$. We will choose the goal parameter of each run to be small enough that the garbage quota of the run immediately above it is not threatened.

The verification proceeds as follows. We can see by induction that each C_i is an i-set. The set L must be a Kraft-Chaitin set based on our careful balancing of the garbage quotas and goals. Furthermore, there is a "golden run" of some procedure P_i which is not cancelled, does not return itself, and for which every run of a procedure of the form Q_i that it starts is either cancelled or returns. This can be seen by assuming that every run of every procedure is either returned or cancelled and obtaining a contradiction.

Finally, we show that A is low for K. We choose a golden run of a procedure P_i and enumerate a Kraft-Chaitin set W as follows. If a run of the type Q_j returns, we add an axiom to W of the form $\langle |\sigma| + g + 1, \tau \rangle$, where the components of this axiom are determined by the elements of the 4-tuple indexing the Q_j run and the goal and garbage quota of P_i. Since we have carefully assigned the garbage quotas and goals for each procedure to be sufficiently small, the total measure of W will be no more than 1, and W will be a Kraft-Chaitin set. Let M_e be the machine for W obtained by the Kraft-Chaitin Theorem. Then we can show that for all τ, $K(\tau) \leq K^A(\tau) + g + e + 1$. If $K_s^A(\sigma) = \tau$, where σ is the shortest description of τ and s is the least stage at which this computation is stable, $Q_{i-1,\sigma,\tau,w}$ must be called. Since the run of P_i is not cancelled, it must return, and at that stage, $\langle |\sigma| + g + 1, \tau \rangle$ will enter W. This gives us $K(\tau) \leq |\sigma| + g + 1$, and since $K^A(\tau) \leq |\sigma| + e$, we have

$$K(\tau) \leq K^A(\tau) + g + e + 1$$

and we are done. □

Corollary 2.4. *A real is K-trivial if and only if it is low for Martin-Löf randomness.*

3. Lowness for recursive randomness

In this section, we present the characterizations of Low(ML,Rec) and Low(Rec) originally obtained by Nies in [23]. In this case, the latter can be obtained easily from the former. The proofs in this section make heavy use of martingales, since this is the simplest way to characterize recursive randomness. In the first proof, we also use one of the characterizations of Low(ML) given in the previous section.

Theorem 3.1. [23] *A real A is in* Low(ML,Rec) *if and only if A is low for K and thus in* Low(ML) *(and K-trivial).*

Proof. As previously noted, if A is low for K, then $A \in$ Low(ML). Since Low(ML) $\subseteq$ Low(ML,Rec), we need only show that if A is in Low(ML,Rec), then A is low for K. We will build a martingale functional L such that if L^A only succeeds on reals that are not Martin-Löf random, then A must be low for K. Since such a real A must be in Low(ML,Rec), we will have the desired inclusion.

The general outline of the proof is as follows. We will begin with an r.e. open set R in the Cantor space with measure strictly less than 1 that contains all reals that are not Martin-Löf random. We will then construct a martingale functional L (a Turing functional such that L^X is a martingale for every oracle X) such that if L^A only succeeds on reals that are not Martin-Löf random, then A must be low for K. To demonstrate that A will be low for K in such a case, we will define a Kraft-Chaitin set W that witnesses this fact by ensuring that for some $c \in \omega$, if $K^{A\upharpoonright n}(\sigma) = m$, then $\langle m + c, \sigma\rangle \in W$. As we build W, we will ensure that it is a Kraft-Chaitin set by balancing the axioms that we allow to enter W against the measure that is enumerated into R at any given stage.

We begin by noting that it is not difficult to find an open set R as described above: simply let $R = \{\sigma \mid (\exists \tau \preceq \sigma)[K(\tau) \leq |\tau| - 1]\}$. In this case, the measure of R is actually less than 2^{-1}, and we can see that any real that is not Martin-Löf random will be contained in R. We now note the following lemma.

Lemma 3.2. *Let d be a martingale that does not succeed on any Martin-Löf random real. Then there are $\sigma \in 2^{<\omega}$ and $c \in \omega$ such that $\sigma \notin R$ and for all $\tau \succeq \sigma$, τ is in R only if $d(\tau) \geq 2^c$.*

This lemma is justified as follows. If it failed, we could define a sequence of strings $\langle\sigma_n\rangle_{n\in\omega}$ starting with the empty string such that σ_{n+1} is a proper extension of σ_n, $d(\sigma_{n+1}) \geq 2^n$, and $\sigma_{n+1} \notin R$. Then d would succeed on $X = \lim \sigma_n$, but, since X cannot be in R, X would have to be Martin-Löf random.

Now we turn to our martingale functional L. We will independently construct a martingale functional L_n for each n that has the value 2^{-n} on any input of length $\leq n$. If we define L to be $\sum_{n\geq 1} L_n$, this will ensure that L will be a rational-valued martingale functional as well.

We enumerate all triples $\delta_n = \langle \tau, a, u\rangle$, where $\tau \in 2^{<\omega}$ and $a, u \in \omega$. If τ and a are witnesses for the above Lemma and $0 < 2^{-u} < 1 - \mu_\tau(R)$, we will define a Kraft-Chaitin set witnessing A's lowness for K. However, since we cannot identify the witnesses τ, a, and u in advance, we must consider all possible triples δ_n. For each δ_n, we will build a sequence of finite trees $\langle T_s\rangle_{s\in\omega}$ in $2^{<\omega}$ such that $T = \lim_s T_s$ exists and such that if δ_n is actually a witness, A is a path in T. As we build T, we build an accompanying Kraft-Chaitin set W such that if $\gamma \in T$ and $K^\gamma(\rho) = r$, then $\langle r + c, \rho\rangle$ will enter W, where $c = n + a + u + 3$.

At each stage s for each triple δ_n, after we determine T_s, we may carry out a *procedure* α to construct our W. Each procedure is a triple of strings $\langle \sigma, \rho, \gamma\rangle$ with certain length conditions. We start α at the least stage s such that $\gamma \in T_s$ and $U_s^\gamma(\sigma) = \rho$, where U is the universal Turing machine we are using to compute prefix-free Kolmogorov complexity. At this point, α will try to make $\langle |\sigma| + c, \rho \rangle$ enter W. However, this must be done with care. For W to be a Kraft-Chaitin set, it must have measure no greater than 1. To ensure this, we will wait to add $\langle |\sigma| + c, \rho \rangle$ to W until a particular clopen set of measure $2^{-(|\sigma|+c)}$ enters R. Once that happens, α will enumerate this tuple into W. We must also take care that the clopen sets associated with different procedures are disjoint so we do not overcount the measure and make $\mu(W)$ greater than 1. To do this, we will build the clopen set for α in pieces that do not overlap with any such set previously chosen and whose measure will be a fixed fraction of $2^{-(|\sigma|+c)}$. When one such set enters R, we assign a new set of a fixed measure less than the remaining part of the allotted $2^{-(|\sigma|+c)}$ and wait until α appears again. If α reappears infinitely often, the clopen set will have the appropriate measure and $\langle |\sigma| + c, \rho \rangle$ will enter W. Otherwise, it will keep away a set of very small measure.

This procedure α will also build the martingale functional L_n by choosing certain strings υ and ensuring that for all reals X extending γ, $L_n^X(\upsilon) \geq a$. It does so by claiming a certain amount of L_n's initial capital, betting it on these chosen strings, and withdrawing it from other strings to ensure that L_n remains a martingale functional. Here, care must be taken to choose strings υ that are not chosen by other procedures and that have not previously been used by α. This will be possible if we choose the υs to be sufficiently long that they only consume a small fraction of the available measure. □

This result gives us a quick proof that the reals that are low for recursive randomness are precisely those that are recursive.

Theorem 3.3. [23] *A real is low for recursive randomness if and only if it is recursive.*

Proof. We begin by observing that Low(Rec) $\subseteq$ Low(ML,Rec) $\subseteq \Delta_2^0$. In [3], Bedregal and Nies showed that for every hyperimmune real A, there is an A-recursive martingale d^A that succeeds on some recursively random real R, so every real that is low for recursive randomness must be hyperimmune free. Since Martin and Miller proved in [22] that the only Δ_2^0 reals that are hyperimmune free are the recursive reals, Low(Rec) is simply the set of recursive reals. □

4. Lowness for Schnorr randomness

In this section, we present characterizations of Low(ML,Schnorr), Low(Rec,Schnorr), and Low(Schnorr). The theorems whose proofs are presented in this section were originally proved by Kjos-Hanssen, Nies, and Stephan [16]; however, several of the techniques they used originated in Terwijn and Zambella's work [34].

First, we mention a notion related to lowness for Schnorr randomness: lowness for Schnorr tests. A real A is said to be low for Schnorr tests if for every Schnorr test relative to A, $\langle V_i^A \rangle_{i\in\omega}$, there is a Schnorr test $\langle U_i \rangle_{i\in\omega}$ such that $\cap_i V_i^A \subseteq \cap_i U_i$. Any real that is low for Schnorr tests will clearly be low for Schnorr, but the converse is not obviously true since there is no universal Schnorr test [29]. Ambos-Spies and Kučera asked in [1] whether these notions were equivalent.

In [34], Terwijn and Zambella characterized the reals that were low for Schnorr tests using the notion of traceability, which is a way of placing bounds on the functions that can be obtained when a particular real is used as an oracle. Later, Kjos-Hanssen, Nies, and Stephan answered Ambos-Spies and Kučera's question affirmatively by showing that these are precisely the reals that are low for Schnorr, and they were able to use the techniques appearing in [34] to describe Low(ML,Schnorr) as well [16]. The characterizations of Low(Schnorr) and Low(Rec,Schnorr) can be obtained easily from the characterization of Low(ML,Schnorr) and a result of Bedregal and Nies [3].

Definition 4.1. A real A is said to be r.e. traceable if there is an order function p, called a bound, such that for all $f \leq_T A$, there is a recursive function r such that for all n, $|W_{r(n)}| \leq p(n)$ and $f(n) \in W_{r(n)}$. A real is said to be recursively traceable if the same statement holds when $W_{r(n)}$ is replaced by $D_{r(n)}$, where $\langle D_m \rangle_{m\in\omega}$ is a canonical ordering of the finite sets (for instance, we may say that D_m contains precisely the natural numbers which are positions in which a 1 occurs in the binary expansion of m).

We can see that a recursively traceable real may be considered to be uniformly hyperimmune free. This notion provides us with a characterization of lowness for Schnorr tests.

Theorem 4.2. [34] *A real is recursively traceable if and only if it is low for Schnorr tests.*

The following proposition will be used in the proof of Theorem 4.2.

Proposition 4.3. [34] *Suppose that A is r.e. (recursively) traceable with respect to a bound function p. Then A is r.e. (recursively) traceable with respect to any bound function q.*

To see this, we note that we can simply make our bound function grow more slowly by considering $g\restriction f(i) \leq_T A$ rather than $g \leq_T A$ for some suitably fast-growing recursive function f.

Theorem 4.4. [16] *A real A is in* Low(ML,Schnorr) *if and only if it is r.e. traceable.*

Proof. Suppose that A is in Low(ML,Schnorr). To prove that A is r.e. traceable, we must show that we can find an r.e. trace for an arbitrary $f \leq_T A$.

We begin by coding f into a Schnorr test relative to A as follows: we define $B_{k,f(k)}$ to be $\cup_k\{[\tau 1^k] \mid |\tau| = f(k)\}$ and let $V_i^f = \cup_{k>i} B_{k,f(k)}$. Without loss of generality, we can assume that this is a Schnorr test relative to A. Since A is in Low(ML,Schnorr), we can further assume that $\cap_i V_i^f \subseteq \cap_i U_i$, where $\langle U_i \rangle_{i\in\omega}$ is a universal Martin-Löf test.

This will allow us to calculate a trace r for f using only the fact that $\cap_i V_i^f \subseteq U_3$. To ensure that each $W_{r(k)}$ has a small enough cardinality, we will only let n enter $W_{r(k)}$ if $B_{k,n} - U_3$ is sufficiently small. Since $\mu(U_3) \leq 4^{-1}$, it will normally be the case that $B_{k,n} - U_3$ has relatively large measure. Therefore, there will not be many n such that $B_{k,n} - U_3$ is small enough, so each $W_{r(k)}$ will contain only a few elements. We must balance this requirement on the size of $W_{r(k)}$ against our need to ensure that r is actually a trace for f. We will be able to do this based on our method of encoding the initial segments $f\restriction n$ into our V_i^fs. The formal proof requires several long measure-theoretic calculations that we omit here.

Now suppose that A is r.e. traceable, and let $\langle V_i^A \rangle_{i\in\omega}$ be a Schnorr test relative to A. For each i, we will let $\langle V_{i,s}^A \rangle_{s\in\omega}$ be a sequence of clopen sets that is uniform in s and i such that $V_i^A = \cup_s V_{i,s}^A$ and $\mu(V_{i,s}^A) > 2^{-i}(1-2^{-s})$ for all i and s. We now let $f \leq_T A$ be a function such that $[S_{f(\langle i,s\rangle)}] = V_{i,s}^A$, where the S_is are the canonical finite sets of elements of $2^{<\omega}$. Since A is r.e. traceable via some function r, we can apply Proposition 4.3 and choose $p(n) = n$ as our bound function.

We will now construct a function $\widehat{r}$ such that $W_{\widehat{r}(\langle i,s\rangle)} \subseteq W_{r(\langle i,s\rangle)}$. Our goal is to make each $W_{\widehat{r}(\langle i,s\rangle)}$ so small that we can use them to build a Martin-Löf test whose intersection contains $\cap_i V_i^A$ and still avoid losing any information in $W_{r(\langle i,s\rangle)}$. To do this, we consider only the elements e of $W_{r(\langle i,s\rangle)}$ such that $2^{-i}(1-2^{-s}) < \mu([S_e]) < 2^{-i}$ and $[S_e] \supseteq [S_d]$ for some $d \in W_{\widehat{r}(\langle i,s\rangle)}$. Then we let

V_i be the union of all $[S_e]$s obtained by considering pairs of the form $\langle i, s\rangle$. The measures of these V_is will be bounded, and we can recursively find a subsequence of them that forms a Martin-Löf test. The intersection of the elements of this Martin-Löf test will contain $\cap_i V_i^A$, so every real that is Martin-Löf random will be Schnorr random relative to A, and $A \in \text{Low}(\mathsf{ML},\mathsf{Schnorr})$. □

Theorem 4.5. [16] *The following are equivalent for a real A.*

(1) *A is in* Low(Rec,Schnorr).
(2) *A is in* Low(Schnorr).
(3) *A is recursively traceable.*

Proof. By Theorem 4.2, a recursively traceable real is low for Schnorr tests, so it must be low for Schnorr and thus in Low(Rec,Schnorr). It will therefore be enough to show that any real in Low(Rec,Schnorr) is recursively traceable.

In [3], Bedregal and Nies showed that Low(Rec,Schnorr) is actually a subset of the hyperimmune-free reals. It is clear that any real A that is hyperimmune free and r.e. traceable is actually recursively traceable, since, given a function $f \leq_T A$ and an r.e. trace r, we can compute the least stage at which $f(n)$ appears in $W_{r(n)}$ recursively and obtain a recursive trace. Finally, since Low(Rec,Schnorr) $\subseteq$ Low(ML,Schnorr), we can use Theorem 4.4 to see that every element of Low(Rec,Schnorr) is recursively traceable. □

5. Lowness for Kurtz randomness

In contrast to the previous section, the characterization of Low(Kurtz) is not presented as a corollary of the characterizations of Low(R,Kurtz) for other classes R. Instead, we present the characterization of Low(Kurtz) and then characterize the potentially larger classes Low(ML,Kurtz), Low(Rec,Kurtz), and Low(Schnorr,Kurtz).

In [10], Downey, Griffiths, and Reid showed that any Schnorr low real is low for Kurtz tests and that every real that is low for Kurtz tests is hyperimmune free using similar techniques to those in [34]. It was hypothesized that Low(Kurtz)=Low(Schnorr); however, Stephan and Yu showed that this was not the case and demonstrated that any real that was hyperimmune free and not DNR was low for Kurtz randomness [33]. Greenberg and Miller have recently shown that the converse is true, completing the characterization [12]. Recall that a function f is *diagonally nonrecursive* (DNR) if for any e such that $\varphi_e(e)$ converges, $f(e) \neq \varphi_e(e)$, and that a real is said to be DNR if it computes such a function. We begin with Stephan and Yu's result.

Theorem 5.1. [33] *If a real is hyperimmune free and not DNR, then it is low for Kurtz randomness.*

Proof. Suppose that A is hyperimmune free and not DNR, and let U^A be a Σ_1^A class of measure 1. We will build a Σ_1^0 class T of measure 1 such that $T \subseteq U^A$.

We begin by observing that U^A must be dense. We can therefore find a function F recursive in A such that, for every n, every string of length n has a proper extension of length $F(n)$ that determines a neighborhood that is entirely contained in U^A and such that the sum of the measures of these neighborhoods exceeds $1 - 2^{-n}$. We can then use the fact that A is hyperimmune free to find a recursive function f such that $f(n+1) > F(f(n))$ for all n. This allows us to compute an A-recursive function g such that $g(n)$ codes a set of finite binary strings of length $f(n+1)$ extending every string of length $f(n)$ whose neighborhoods are contained in U^A and have measure within 2^{-n} of 1. We now make use of the following result of Kjos-Hanssen, Merkle, and Stephan.

Proposition 5.2. [14] *If A is hyperimmune free and not DNR, then for every $g \leq_T A$, there are recursive functions h and $\widehat{h}$ such that*

$$(\forall n)(\exists m \in \{n, n+1, \dots, \widehat{h}(n)\})[h(m) = g(m)].$$

We can now define T to be

$$\{X \mid (\exists n)(\forall m \in \{n, n+1, \dots, \widehat{h}(n)\})[X \restriction f(m+1) \in h(m)]\}.$$

Our definition of g lets us see easily that this set is dense, and it is clearly defined in a Σ^0_1 way. Finally, if $X \in T$, there are n and m such that $m \in \{n, n+1, \dots, \widehat{h}(n)\}$ and $h(m) = g(m)$, so X extends $g(m)$. Therefore, $T \subseteq U^A$. □

We now turn our attention to the converse implication, first proven by Greenberg and Miller. Kurtz showed in his thesis that every hyperimmune degree contains a Kurtz random real, so every real that is low for Kurtz randomness is necessarily hyperimmune free [18]. Therefore, it is sufficient to show that a real that is low for Kurtz randomness cannot be DNR. Recently, Miller has obtained a shorter proof of this result which involves an argument similar to the proof given here of Theorem 7.3 [2].

Greenberg and Miller began by proving the corresponding result for the notion of lowness for Kurtz tests. Lowness for Kurtz tests is analogous to lowness for Schnorr tests, which we mentioned in Section 4.

Theorem 5.3. [12] *If a real is low for Kurtz tests, then it is not DNR.*

Proof. This proof is heavily computational. Greenberg and Miller first showed that any real that is low for Kurtz tests is not DNR. To do this, they introduced the notion of svelte trees.

Definition 5.4. A finite subtree T of $\omega^{<\omega}$ is said to be k-svelte if there is a sequence $\langle S_{k+i} \rangle_{i \in \omega}$ of subsets of T such that the following three conditions hold for a particular fixed increasing sequence of natural numbers $\langle n_m \rangle_{m \in \omega}$.

(1) $S_m \in T \cap \omega^{n_m}$.
(2) $|S_m| \leq 2^{m-(k+1)}$.
(3) Every leaf of T extends a string in $\cup_m S_m$.

Such a tree can be covered by so few basic clopen sets that none of its paths can be made to be DNR.

To show that a real that is low for Kurtz tests is not DNR, we show that if a function f is DNR, there must be a Π^f_1 class of measure 0 that is not contained in any Π^0_1 class of measure 0. We define an operator that takes an $f \in \omega^{\leq \omega}$ to a closed set $P^f \subseteq 2^\omega$ in such a way that $\mu(P^f) = 0$ for all f and that P^f and P^g will be sufficiently independent for $f \neq g$. This allows us to define P^T as the union of P^σ for all leaves σ of T if T is finite and as the union of P^f for all branches f in T if T has no dead ends.

We then show that any finite tree T such that $\mu(P^T) \leq 2^{-(k+1)}$ is k-svelte and use this information to work with infinite trees $T \subseteq \omega^{<\omega}$. Such a tree is defined to be full-by-finite if there is a finite tree S with each leaf at some level n_m such that $T = S \cup \{\sigma \in \omega^{<\omega} \mid \sigma \text{ extends a leaf of } S\}$. We now consider a full-by-finite tree T such that $\mu(P^T) \leq 2^{-(k+1)}$ and suppose that its full-by-finiteness is witnessed by S. Such an S can be seen to be k-svelte. We can then show that every clopen $C \subseteq 2^\omega$ has an associated full-by-finite tree T such that $[T] = \{f \in \omega^\omega \mid P^f \subseteq C\}$.

Finally, we build a partial recursive function ψ such that for the k^{th} Π^0_1 class Q_k, if we can see that $\mu(Q_k, s) < 2^{-(k+1)}$ at some stage s, we will compute a finite tree S that is k-svelte and whose upwards closure is the tree of paths f such that $P^f \subseteq Q_{k,s}$. This gives us witnesses $\langle S_{k+1}, \ldots \rangle$ to S's k-svelteness. We then define ψ in such a way that for any string $\sigma \in S_m$, σ is not a DNR string by using the Recursion Theorem. Since every f in this tree of paths extends such a σ, no such f can be DNR itself. □

This leads easily to the following theorem, which finishes the characterization of Low(Kurtz).

Theorem 5.5. [12] *Any real that is DNR is not low for Kurtz randomness.*

Proof. First, we show that if $f \in \omega^\omega$ and a nonempty clopen subclass of P^f is covered by a null Π^0_1 class, then P^f itself is, too. We can then see from the previous theorem that if f is DNR, P^f is not contained in a null Π^0_1 class, so neither are any of its clopen subsets. We can then build a Kurtz random real contained in P^f by extensions that avoid all null Π^0_1 classes. □

Finally, we present characterizations of Low(ML,Kurtz), Low(Rec,Kurtz), and Low(Schnorr,Kurtz).

Theorem 5.6. [6, 12] *A real is in* Low(ML,Kurtz) *if and only if it is not DNR.*

Proof. Kjos-Hanssen showed in [6] that if A is a real that does not compute a DNR function, every Π^A_1 null class P is contained in the intersection of a Martin-Löf test. This is enough since lowness for Martin-Löf tests coincides with lowness for Martin-Löf randomness. To do this, he used an A-recursive, nested sequence $\langle C_i \rangle_{i\in\omega}$ of clopen sets such that $\mu(C_i) = 2^{-i}$ and $P = \cap_i C_i$. Since A is not DNR, there are infinitely many n such that $\varphi_n(n)$ is a code for C_n. We can then define a Martin-Löf test $\langle V_i \rangle_{i\in\omega}$ such that V_i is the union of all C_n such that $n > i$ and $\varphi_n(n)$ codes C_n. Clearly, $P \subseteq \cap_i V_i$.

Greenberg and Miller proved the other direction of this theorem by using the machinery from the previous theorems mentioned here. Given a universal Martin-Löf test $\langle U_i \rangle_{i\in\omega}$, we can build a finite k-svelte tree whose upward closure is the tree of paths f such that $P^f \subseteq U_{k+1,s}$ when k is enumerated into $0'$ at stage s and ensured that none of the paths through the tree are DNR. Then we can see that if f is DNR and $P^f \subseteq \cap_k U_k$, it must be the case that $0' \leq_T f$. Since there is a Martin-Löf random set $R \leq_T 0'$, no degree $\geq_T 0'$ can be in Low(ML,Kurtz), and the theorem is proved. □

The classes Low(Rec,Kurtz) and Low(Schnorr,Kurtz) coincide and strictly contain Low(ML,Kurtz).

Theorem 5.7. [12] *The classes* Low(Rec,Kurtz) *and* Low(Schnorr,Kurtz) *are both precisely the class of reals that are not high or DNR.*

Proof. Since Low(Schnorr,Kurtz) $\subseteq$ Low(Rec,Kurtz) $\subseteq$ Low(ML,Kurtz), by the previous theorem, all of these classes are disjoint from DNR. Since every high degree contains a recursively random real [25], we can also see that Low(Schnorr,Kurtz) and Low(Rec,Kurtz) are disjoint from the high degrees, so we simply need to show that every real A that is not high or DNR is in Low(Schnorr,Kurtz). To do this, we need Proposition 5.2 again.

Given a null Π^A_1 class, we consider the sequence $\langle C_i \rangle_{i\in\omega}$ of the previous proof. We then use the recursive function h given by Kjos-Hanssen, Merkle, and Stephan's result to "pad" the Martin-Löf test built in the previous proof and convert it to a Schnorr test, thus proving the theorem. □

6. Lowness for weak 2-randomness

We now turn our attention to weak 2-randomness, the strongest of the randomness notions discussed in this paper. This is the only notion for which two classes remain uncharacterized: Low(W2R,Schnorr) and Low(W2R,Kurtz). In this section, we will present characterizations of Low(W2R), Low(W2R,ML), and Low(W2R,Rec).

The first results in this area come from Downey, Nies, Weber, and Yu [9], who showed that Low(W2R,ML)=Low(ML). This provides a partial characterization of Low(W2R), since Low(W2R) $\subseteq$ Low(W2R,ML). Later, Kjos-Hanssen, Miller, and Solomon showed that Low(ML) $\subseteq$ Low(W2R), which completed the characterization [15]. There is also a proof of this result by Nies using the "golden run" argument described in Section 2 in [24]. Finally, a variation by Nies on the proof of Theorem 3.1 provides a characterization of Low(W2R,Rec) [24]. We begin with the results of Downey, Nies, Weber, and Yu.

Theorem 6.1. [9] *The reals that are low for Martin-Löf randomness are precisely the elements of* Low(W2R,ML)*.*

Proof. We can see immediately that Low(ML) $\subseteq$ Low(W2R,ML). We now suppose that a real A is not low for Martin-Löf randomness. This allows us to apply the following result based on a theorem by Stephan appearing in [23].

Proposition 6.2. *Suppose that A is not low for Martin-Löf randomness and that β and γ are rationals such that $\beta < \gamma < 1$. For every r.e. open set V and string σ, if $\mu_\sigma(V) \leq \beta$, then there is a string τ such that $K^A(\tau) \leq |\tau| - 1$ and $\mu_{\sigma\tau}(V) \leq \gamma$.*

We define a sequence of strings $\langle \sigma_i \rangle_{i\in\omega}$ such that for each i, $K^A(\sigma_i) \leq |\sigma_i| - 1$. By a result of Merkle that appears in [24], the concatenation of these σ_is will not be Martin-Löf random relative to A.

To build this sequence, we take an enumeration of all potential generalized Martin-Löf tests. If the e^{th} test, $\langle V_i^e \rangle_{i\in\omega}$, is actually a generalized Martin-Löf test, we will define a number n_e and ensure that the concatenation of the σ_is is not in $V_{n_e}^e$. At each stage e, we will define the string σ_e. We let U_e be the union of those $V_{n_i}^i$ such that n_i has been defined and $i < e$. We can ensure that $\mu_{\sigma_0\dots\sigma_{e-1}}(U_e) \leq 1 - 2^{-e}$ inductively, and this will allow us to find an appropriate σ_e via Proposition 6.2. □

Corollary 6.3. *Every real that is low for weak 2-randomness is low for Martin-Löf randomness.*

Theorem 6.4. [15, 24] *Every real that is low for Martin-Löf randomness is low for weak 2-randomness.*

Proof. Kjos-Hanssen, Miller, and Solomon's proof uses a reducibility originally defined in [23], $\leq_{LR}$, which is based on lowness for Martin-Löf randomness. Given two reals A and B, we say that $A \leq_{LR} B$ if every real that is Martin-Löf random relative to B is Martin-Löf random relative to A. Clearly, if A is low for Martin-Löf randomness, then $A \leq_{LR} 0^\omega$. We recall that every element of Low(ML) is Δ_2^0 and thus $\leq_T 0'$, which allows us to apply the following theorem from the same paper.

Theorem 6.5. *The following are equivalent for any two reals A and B.*

(1) $A \leq_T B'$ *and* $A \leq_{LR} B$*.*

(2) *Every* Π^A_1 *class has a* Σ^B_2 *subclass of the same measure.*
(3) *Every* Σ^A_2 *class has a* Σ^B_2 *subclass of the same measure.*

This theorem lets us see that every Σ^A_2 class has a Σ^0_2 subclass of the same measure, so every Π^A_2 class of measure 0 is a subclass of some Π^0_2 class of measure 0. Therefore, every generalized Martin-Löf test relative to A is contained in an unrelativized generalized Martin-Löf test, and we are done. □

Corollary 6.3 and Theorem 6.4 allow us to see that the reals that are low for weak 2-randomness are precisely those that are low for Martin-Löf randomness.

Finally, we discuss the characterization of Low(W2R,Rec).

Theorem 6.6. [24] *The class* Low(W2R,Rec) *is precisely* Low(ML).

Proof. As previously mentioned, this is a modification of the proof of Theorem 3.1. Instead of triples of the form $\delta_m = \langle \tau, a, u\rangle$, we use 5-tuples of the form $\delta_m = \langle \tau, a, V, q, \epsilon\rangle$, where $\tau \in 2^{<\omega}$, $a \in \omega$, V is an r.e. open set, and q and ϵ are positive dyadic rationals such that $q + \epsilon \leq 1$. For each witness δ_m, we define R to be the set $\{\sigma \succeq \tau \mid \mu_\sigma(V) \geq q + \epsilon\}$, and we let u be the least natural number such that $\frac{q}{q+\epsilon} < 1 - 2^{-u}$. Now a procedure for δ_m will be active only at stages s such that $\mu_\tau(R_s) < 1 - 2^{-u}$. Finally, instead of Lemma 3.2, we use the following lemma.

Lemma 6.7. *Suppose that* d *is a martingale that only succeeds on reals that are not weakly 2-random, and list the quintuples* $\delta_m = \langle \sigma, c, V, q, \epsilon\rangle$ *where* $\sigma \in 2^{<\omega}$, $c \in \omega$, V *is an r.e. open set, and* q *and* ϵ *are positive dyadic rationals such that* $q + \epsilon \leq 1$. *Then there is a* δ_m *such that* $\mu_\sigma(V) \leq q$ *and for all* $\tau \succeq \sigma$, *if* $d(\tau) \geq 2^c$, *then* $\mu_\tau(V) \geq q + \epsilon$.

Given these changes, the argument proceeds as before. □

7. Highness for randomness notions

We now turn our attention to highness for pairs of randomness notions. This concept was originally introduced by Franklin, Stephan, and Yu in [11] and has been further explored by Barmpalias, Miller, and Nies in [2] for relativized versions of several of the randomness notions discussed here. Almost all of the results presented in this section appear in [11]; the exception is the characterization of High(ML,W2R), which was determined by Miller [21] and appears in [2].

High(Schnorr,W2R), High(Schnorr,ML), and High(Schnorr,Rec) all turn out to be the same class of reals, and the proofs are essentially identical.

Theorem 7.1. [11] *The classes* High(Schnorr,W2R), High(Schnorr,ML), *and* High(Schnorr,Rec) *are all equal to the class of reals that compute the halting problem:* $\{X \in 2^\omega \mid X \geq_T 0'\}$.

Proof. One direction of the proof of this theorem is reasonably simple. If $A \geq_T 0'$, then any real in $\mathsf{Schnorr}^A$ is actually weakly 2-random, so it is certainly recursively random and Martin-Löf random. The other direction of the proof of this theorem is based on the following result, which also appears in [11].

Proposition 7.2. *If* A *and* B *are reals and* $A \not\geq_T 0'$, *then there is a real* R *such that the following three conditions hold.*

(1) $B \leq_T R$.
(2) R *is not recursively random.*

(3) *R is Schnorr random relative to A.*

Given this proposition, the theorem will follow easily. If $A \not\geq_T 0'$ and we let $B = A$, then the proposition tells us that there is a real that is Schnorr random relative to A that is not recursively random. Therefore, it will not be Martin-Löf random or weakly 2-random, either.

Proof of Proposition 7.2. We will construct R explicitly in intervals. Some of these intervals will be used to code B into R, and others will be used to ensure that no A-recursive martingale is too successful on R.

We fix an enumeration of the halting problem, K. The lengths of the intervals will be defined by a recursive, injective enumeration of all pairs $\langle a_m, b_m \rangle$ such that $a_m > 0$ and either $b_m = 0$ or some element below a_m is enumerated into K at stage b_m. We then divide ω into consecutive intervals I_m of length $3a_m + 1$ and let d be a weighted sum of all total A-recursive martingales. We now define an infinite A-recursive set E of elements using a K-recursive function maximizing c_K and use this set to divide the intervals I_m into two infinite classes. For all intervals I_m of the first type, we define R on I_m so it is not 0 on all of the least $2a_m$ elements of I_m and so d will grow on I_m by a very small factor. Otherwise, we define R to be 0 on all of the least $2a_m$ elements of I_m and let the next $a_m + 1$ elements of R be the first $a_m + 1$ elements of B.

It is clear that $B \leq_T R$: to compute the n^{th} bit of B, we find some $a_m \geq n+1$ such that the first $2a_m$ bits of R on I_m are 0 and then read off the n^{th} bit of B. To see that R is recursively random, we build a recursive martingale that succeeds on it. To do this, we make our martingale bet that R will be 0 on the first $2a_m$ bits of every interval and then bet nothing on the other $a_m + 1$ bits. The losses of this martingale will be bounded, while it will gain a certain fixed amount infinitely often, ensuring that R will not be recursively random. Finally, we have chosen R so that d does not gain too much on enough intervals I_m, so we can see that R is not Schnorr random relative to A. □

As argued above, this is enough to prove our theorem. □

We now consider the class High(ML,W2R). The following proof appears in [2].

Theorem 7.3. [2] *The class* High(ML,W2R) *is the class of all reals A such that there is no function recursive in $0'$ that is DNR relative to A; i.e., there is no $f \leq_T 0'$ such that $\varphi_e^A(e) \neq f(e)$ whenever $\varphi_e^A(e)$ is defined.*

Proof. We will first show that if A is in High(ML,W2R), then no function recursive in $0'$ is DNR relative to A. This proof involves another class of reals: High(ML,Kurtz[$0'$]), where Kurtz[$0'$] is the class of reals that are Kurtz random relative to $0'$. We begin by observing that since every $\Pi_1^{0,0'}$ class is a Π_2^0 class, High(ML,W2R) is a subset of High(ML,Kurtz[$0'$]). Now we must show that for each element A of High(ML,Kurtz[$0'$]), there is no function f recursive in $0'$ that is DNR relative to A.

We suppose that there is a function f as described above. We will show that $0'$ computes an infinite subset D of an set that is Martin-Löf random with respect to A. This set D will be a member of a null $\Pi_1^{0,0'}$ class and thus not Kurtz random relative to $0'$, but it will also be Martin-Löf random with respect to A, giving us a contradiction.

We begin by taking a nonempty $\Pi_1^{0,A}$ class Q of sets that are Martin-Löf random relative to A and observing that, by a result of Kučera, if we are given a nonempty $\Pi_1^{0,A}$ class $P \subseteq Q$, we can compute a k such that $2^{-k} < \mu(P)$ uniformly from an index for P [17]. Then we use f to compute an increasing sequence $\langle d_n \rangle_{n\in\omega}$ such that the $\Pi_1^{0,A}$ class P_n of all elements of Q containing $\{d_0, \dots, d_n\}$ is nonempty for every n. At each stage, we will choose d_{n+1} to avoid the set $G_n = \{m \mid \forall Z \in P_n(Z(m) = 0)\}$. We then use the fact that f is DNR relative to A and recursive in $0'$ to show that we can compute the sequence $\langle d_n \rangle_{n\in\omega}$ from $0'$. We can then see that the intersection of the P_ns is a nonempty $\Pi_1^{0,0'}$ class of measure 0, and it is clearly contained in Q. Therefore, any element of their intersection must be Martin-Löf random relative to A but not Kurtz random relative to $0'$.

Now suppose that there is no function recursive in $0'$ that is DNR relative to A, and let $\langle V_n \rangle_{n\in\omega}$ be a sequence of Σ_1^0 classes whose measure converges to 0 as n increases. It will be enough to build a Martin-Löf test relative to A, $\langle U_n^A \rangle_{n\in\omega}$, such that $\cap_n V_n \subseteq \cap_n U_n^A$.

For any Σ_1^0 class V and positive rational ϵ, we will let $(V)_\epsilon$ denote the Σ_1^0 class obtained from V by enumerating it just as long as its measure is less than or equal to ϵ. We also note that $0'$ can compute a function f such that $\mu(V_{f(k)}) \leq 2^{-k}$ for all $k \in \omega$. We now define U_n^A to be $\cup_{k>n}(V_{\varphi_k^A(k)})_{2^{-k}}$ for each n. (We will say that $V_{\varphi_k^A(k)} = \emptyset$ if $\varphi_k^A(k)$ diverges.) Since, by our assumption, $0'$ does not compute a function that is DNR relative to A, there are infinitely many k such that $f(k) = \varphi_k^A(k)$. Therefore, U_n^A covers $\cap_n V_n$ for all n. By definition, $\mu(U_n^A) \leq 2^{-n}$, so $\langle U_n^A \rangle_{n\in\omega}$ is a Martin-Löf test relative to A that covers $\cap_n V_n$, and A is an element of High(ML,W2R). □

The classes High(Kurtz,R) are identical for all other R discussed in this article.

Theorem 7.4. [11] *The classes* High(Kurtz,Schnorr)*,* High(Kurtz,Rec)*,* High(Kurtz,ML)*, and* High(Kurtz,W2R) *are all equal to the empty set.*

Proof. Given any A, we can construct a real $R \leq_T A'$ that is Kurtz random relative to A and not Schnorr random. This will, of course, ensure that R is not random with respect to any stronger randomness notion, either.

To do this, we note that we can choose an A'-recursive sequence of numbers $\langle a_i \rangle_{i\in\omega}$ in such a way that R will always be Kurtz random relative to A regardless of the values of R on the intervals $2^{a_m} \leq n < 2^{a_n+1}$. If we let R be constant on each of these intervals, R cannot be Schnorr random, and we are done. □

Finally, we present the partial results obtained on the class High(Rec,ML). We first observe that a natural class of Turing degrees, the PA-complete degrees, is contained in High(Rec,ML). Recall that the PA-complete degrees are those Turing degrees that compute a complete extension of Peano Arithmetic.

Proposition 7.5. [11] *If a real A is PA-complete, then $A \in$* High(Rec, ML)*.*

Proof. We begin by recalling that $A \in$ High(Schnorr,Rec) if and only if $A \geq_T 0'$. We also note that if A is PA-complete, the universal r.e. martingale is majorized by an A-recursive martingale. Therefore, if a real R is recursively random relative to A, neither the A-recursive martingale above nor the universal r.e. martingale succeeds on R, so R must be Martin-Löf random. □

While the above provides a partial characterization, a reversal is not known. However, the following partial result appears in [11].

Theorem 7.6. *If $A \in$* High(Rec, ML)*, then there is a Martin-Löf random real $R \leq_T A$.*

Proof. We prove the contrapositive and begin by supposing that A does not compute any Martin-Löf random real. We will show that A is not in High(Rec,ML) by constructing a real that is not Martin-Löf random that is still recursively random relative to A.

To do this, we will build a function $F \leq_T A'$ such that $A \oplus F$ has high Turing degree relative to A and computes no Martin-Löf random real. By relativizing a result of Nies, Stephan, and Terwijn [25], we can see that there will be a real $Q \leq_T A \oplus F$ that is recursively random relative to A, but Q will not be Martin-Löf random.

To build F, we use a universal r.e. martingale and an injective enumeration of the indices of partial A-recursive functions $\langle e_i \rangle_{i \in \omega}$ such that for all k, there is an $n \leq k$ such that $\varphi^A_{e_k}(n)$ is undefined. This enumeration will be recursive in A'. The function F is built in intervals in such a way that the universal martingale will take on larger and larger values on each interval, while information about A'' is coded into $(A \oplus F)'$. We can use the fact that A does not compute a Martin-Löf random real to ensure that the algorithm does not terminate and F is total. The latter will guarantee the highness of $A \oplus F$ relative to A, while the former will ensure that no Martin-Löf random real will be computable from $A \oplus F$. □

On the other hand, we can see from the following theorem that this result does not lead to a full characterization.

Theorem 7.7. [11] *There is a real A that is Martin-Löf random that is not in* High(Rec,ML).

Proof. This proof relies on a result in [5]. In this paper, Cholak, Greenberg, and Miller proved that there is an incomplete r.e. set B and an almost everywhere dominating (a.e.d.) function f such that $f \leq_T B$. Recall that an a.e.d. function is one that, for a subclass of the Cantor space of measure 1, dominates every function that is recursive relative to a member of that subclass. We use this function to construct a martingale d recursive in B. To do so, we construct a martingale d^E for every real E by using f as a bound on the convergence speed and use of the associated betting strategy and then integrate over all reals E. We can then build a real R recursively in B on which this martingale is not successful. Since B is r.e. and Turing incomplete, R cannot be Martin-Löf random.

Finally, we show that R is recursively random relative to every member of a class of measure 1. To do so, we assume that this is not the case and note that there must be a martingale d_1 such that d_1^A succeeds on R for some set of oracles A that does not have measure 0. Since f dominates all A-recursive functions, we can show that our original martingale, d, must succeed on R, which gives us a contradiction.

Since R is recursively random relative to every member of a class of measure 1, it must be recursively random relative to a Martin-Löf random real. This Martin-Löf random real will not be in High(Rec,ML), and the theorem is proven. □

Therefore, the question remains open for the reals that compute a Martin-Löf random real but not a complete extension of PA.

8. Conclusions

Summaries of the known characterizations of the classes Low(R,P) and High(R,P) appear in Tables 1 and 2 below. The entry in row R and column P represents the characterization of Low(R,P) in Table 1 and High(R,P) in Table 2. A blank box in the table indicates that the characterization of lowness (or highness) for that particular pair of randomness notions is trivially $\emptyset$ (or 2^ω), while a question mark indicates that no full characterization of that set of reals exists.

Table 1. Low(R,P)

	W2R	ML	Rec	Schnorr	Kurtz
W2R	K-trivial	K-trivial	K-trivial	?	?
ML		K-trivial	K-trivial	r.e. traceable	non-DNR
Rec			recursive	recursively traceable	nonhigh, non-DNR
Schnorr				recursively traceable	nonhigh, non-DNR
Kurtz					hyperimmune-free, non-DNR

Table 2. High(R,P)

	Kurtz	Schnorr	Rec	ML	W2R
Kurtz		$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Schnorr			$A \geq_T 0'$	$A \geq_T 0'$	$A \geq_T 0'$
Rec				?	?
ML					$\mathcal{D}$

In Table 2, $\mathcal{D}$ represents the class of reals A such that there is no function f recursive in K that is DNR relative to A.

It is interesting to note that while these five randomness notions give rise to the chain of classes of Theorem 1.8, the basic lowness notions Low(R) do not. In fact, Low(ML), Low(Rec), and Low(Schnorr) only overlap pairwise on the recursive reals. However, we can create such chains by considering the classes Low(R,P) or High(R,P) for a fixed R or P. Note that in Table 1, the classes form increasing chains going up the columns and from left to right and that in Table 2, the classes form increasing chains going down the columns and from right to left.

It should be noted that similar work has been done on lowness for genericity notions and highness for pairs of genericity notions. While one can argue that some results on this subject have been presented here, since Kurtz randomness can reasonably be considered to be a genericity notion, there are further results in this area. The reader is referred to [36] and [33] for work on lowness for genericity notions and to [11] for work on highness for pairs of genericity notions.

We conclude with some open questions.

Question 8.1. [12] Characterize Low(W2R,Schnorr) and Low(W2R,Kurtz).

Question 8.2. [11] (Fully) characterize High(Rec,ML) and High(Rec,W2R).

Question 8.3. [1] Can Low(ML) be characterized in a recursion-theoretic way that is not directly related to a randomness notion?

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