## HYPERIMMUNE-FREE DEGREES AND SCHNORR TRIVIALITY

JOHANNA N.Y. FRANKLIN
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
2, SCIENCE DRIVE 2
SINGAPORE 117543

Abstract. We investigate the relationship between lowness for Schnorr randomness and Schnorr triviality. We show that a real is low for Schnorr randomness if and only if it is Schnorr trivial and hyperimmune free.

### 1. Introduction

There are three different approaches to algorithmic randomness. In the first approach, the unpredictability approach, we say that a real is random if its $(n+1)$st bit cannot be predicted given its first $n$ bits. The second approach is one of measure: we say a real is random if it does not belong to any effectively presented null class. Finally, we may also call a real random if has high initial-segment complexity; i.e., its initial segments cannot be described in a simple way.

Of course, each of these notions is relative to the type of algorithm used to generate the null set, predict the real's bits, or describe its initial segments. We will begin by describing Martin-Löf randomness and Schnorr randomness. Then we will be able to approach the main question of this paper: what does it mean for a real to be "far from random," and if there are multiple definitions, do they coincide?

There are three ways for a real $A$ to be "far from random" that we will consider. One of them is simply that if $A$ is used as an oracle, it is not powerful enough to make a random set appear nonrandom. The second is based on the measure-theoretic approach to randomness. Here, if $A$ is used as an oracle for the presentation of the null class, it will not make a random set appear nonrandom. The third way is based on the initial-segment complexity approach. In this case, a real is "far from random" if its initial-segment complexity is no more than that of a recursive real.

All of these definitions coincide for Martin-Löf randomness [11], but this result is far from obvious. In this paper, we show that this situation does not obtain for Schnorr randomness and provide a precise description of the relationship between three of these notions: they turn out to be equivalent in the hyperimmune-free degrees. At the end, we will present additional evidence that suggests, when combined with these results, that the framework in which the Schnorr trivial reals should be studied is the *tt*-degrees rather than the Turing degrees.

1.1. **Terminology and definitions.** Most of the notation is standard and follows Soare [14]. We will consider real numbers to be elements of $2^\omega$ and Turing machines to be recursive functions from $2^{<\omega}$ to $2^{<\omega}$.

We will use $\mu$ to denote Lesbesgue measure throughout the paper, and for any $\sigma \in 2^{<\omega}$, we will say that $\mu(\sigma) = \mu([\sigma]) = \frac{1}{2^{|\sigma|}}$. If $S \subseteq 2^{<\omega}$ is a prefix-free set of binary strings, we let $\mu(S) = \sum_{\sigma \in S} \mu(\sigma) = \sum_{\sigma \in S} \frac{1}{2^{|\sigma|}}$. We will often consider the measure of the domain of a Turing machine, but never the range. Therefore, we will simply write $\mu(M)$ for $\mu(\text{dom}(M))$. It should be noted that if $M$ is a prefix-free Turing machine and we

This material comprises part of the author's Ph.D. thesis which was written under the supervision of Theodore A. Slaman at the University of California, Berkeley. The author acknowledges the support of a National Science Foundation Graduate Research Fellowship and thanks her advisor and Leo Harrington for many insightful conversations.

list the elements of $M$ as $\langle \tau_i, \sigma_i \rangle$, we can see that $\mu(M) = \sum_i \frac{1}{2^{|\tau_i|}}$. We will assume from this point on that all Turing machines we consider are prefix free.

As in [4], we will use $K$ to denote prefix-free Kolmogorov complexity. In this paper, we will generally not be considering prefix-free Kolmogorov complexity with respect to a universal machine, but Kolmogorov complexity with respect to some other particular Turing machine. This will make the following notation necessary.

**Definition 1.1.** Let $M$ be a Turing machine, and let $\sigma \in 2^{<\omega}$. The *prefix-free Kolmogorov complexity of $\sigma$ with respect to $M$* is $K_M(\sigma) = \min\{|\tau| \mid M(\tau) = \sigma\}$.

If no machine is specified in the subscript, a universal Turing machine is being used.

It is clear that the measure of each Turing machine's domain is a recursively enumerable real; i.e., effectively approximable from below. However, we will generally wish to consider the restricted class of Turing machines that satisfy the following condition.

**Definition 1.2.** A Turing machine $M$ is *computable* if the measure of its domain is a recursive real.

These definitions allow us to define the two types of randomness that we will discuss: Martin-Löf and Schnorr randomness.

**Definition 1.3.** A *Martin-Löf test* is a uniformly r.e. sequence $\langle V_i \rangle_{i \in \omega}$ of $\Sigma^0_1$ classes such that $\mu(V_i) \leq \frac{1}{2^i}$. A real $A$ is *Martin-Löf random* if for all Martin-Löf tests $\langle V_i \rangle_{i \in \omega}$, $A \notin \cap_{i \in \omega} V_i$.

Schnorr proved that this measure-theoretic definition is equivalent to an initial-segment complexity definition.

**Theorem 1.4.** [13] *$A$ is Martin-Löf random if and only if* $(\exists c \in \omega)(\forall n \in \omega)[K(A \upharpoonright n) \geq n - c]$.

Now we turn our attention to Schnorr randomness. Here, we ask that the measure of each Martin-Löf test be recursive instead of simply recursively enumerable.

**Definition 1.5.** A Martin-Löf test is a *Schnorr test* if $\mu(V_i) = \frac{1}{2^i}$ for all $i$. A real $A$ is *Schnorr random* if for all Schnorr tests $\langle V_i \rangle_{i \in \omega}$, $A \notin \cap_{i \in \omega} V_i$.

There is no universal Schnorr test. Therefore, we cannot use a universal Turing machine to define Schnorr randomness in terms of computational complexity. Instead, we must quantify over all computable Turing machines.

**Theorem 1.6.** [5] *A real $A$ is Schnorr random if and only if* $(\forall M \text{ comp.})(\exists c \in \omega)(\forall n \in \omega)[K_M(A \upharpoonright n) \geq n - c]$.

1.2. **Previous work.** As mentioned before, there are three primary ways of describing a real as "far from random." In two cases, we show that the real has no power as an oracle when the randomness of other reals is in question, and in the third, we show that its initial-segment complexity is no greater than that of a recursive real. Here we present three different approaches that each capture one of these concepts. We let $\mathcal{R}$ represent a notion of randomness such as Martin-Löf randomness or Schnorr randomness.

**Definition 1.7.** A real $A$ is *low for $\mathcal{R}$* if $\mathsf{R} = \mathsf{R}^A$, where $\mathsf{R}$ stands for the class of reals that are $\mathcal{R}$-random.

**Definition 1.8.** A real $A$ is *low for $\mathcal{R}$-tests* if for every $\mathcal{R}$-test $\langle U_i^A \rangle_{i \in \omega}$ relative to $A$, there is an $\mathcal{R}$-test $\langle V_i \rangle_{i \in \omega}$ such that $\cap_{i \in \omega} U_i^A \subseteq \cap_{i \in \omega} V_i$.

The third approach, *$\mathcal{R}$-triviality*, cannot easily be defined formally at this level of generality. Each randomness notion $\mathcal{R}$ with an initial-segment complexity definition has an associated reducibility $\leq_{\mathcal{R}}$ used to indicate the relative initial-segment complexity of two reals. We say that a real $A$ is $\mathcal{R}$-trivial if $A \leq_{\mathcal{R}} 0^\omega$; that is, if $A$'s initial-segment complexity is no more than that of a recursive real with respect to this particular reducibility. We will formalize this approach later for the particular case of Schnorr triviality.

Clearly, if $A$ is low for $\mathcal{R}$-tests, it is low for $\mathcal{R}$. However, it is not obvious that either lowness notion is related to the triviality notion.

Nies has shown that in the case of Martin-Löf randomness, these notions describe the same set of reals [11]. Furthermore, in the same paper, he showed that this set of reals is easy to describe: it is a $\Sigma^0_3$ ideal in the $\Delta^0_2$ degrees that is generated by its recursively enumerable members. However, the situation is much more complicated for Schnorr trivial reals, reals that are low for Schnorr tests, and reals that are low for Schnorr randomness.

Downey, Griffiths, and Laforte developed the following characterization of Schnorr triviality in [3], based on a notion of initial-segment complexity for Schnorr randomness previously defined in [5].

**Definition 1.9.** [3] We say that $A \leq_{Sch} B$ if for every computable Turing machine $M$, there is a computable Turing machine $M'$ and a constant $c \in \omega$ such that $(\forall n \in \omega)[K_{M'}(A\upharpoonright n) \leq K_M(B\upharpoonright n) + c]$. Therefore, a real $A$ is *Schnorr trivial* ($A \leq_{Sch} 0^\omega$) if the following statement holds.

$$(\forall M \text{ comp.})(\exists M' \text{ comp.})(\exists c \in \omega)(\forall n \in \omega)[K_{M'}(A\upharpoonright n) \leq K_M(0^n) + c]$$

Downey, Griffiths, and Laforte have proved that there is a Turing complete Schnorr trivial real, but that there is an r.e. degree that contains no Schnorr trivial reals [3]. While this shows that the Schnorr trivial Turing degrees are not downward closed, they also proved that the Schnorr trivial *tt*-degrees are downward closed [3].

The work done to date with reals that are low for Schnorr randomness and those that are low for Schnorr tests has produced an entirely degree-theoretic characterization. Here we let $D_n$ denote the $n$th canonical finite set.

**Definition 1.10.** A set $A$ is *recursively traceable* if there is a recursive, increasing, unbounded function $p : \omega \longrightarrow \omega$ as follows.

$$(\forall f \leq_T A)(\exists r : \omega \longrightarrow \omega \text{ rec.})(\forall n \in \omega)[f(n) \in D_{r(n)} \text{ and } |D_{r(n)}| \leq p(n)]$$

A recursive, increasing, unbounded function such as $p$ is called an *order function.*

We may therefore think of the recursively traceable reals as those that are uniformly hyperimmune free. We say that $r$ is a *recursive trace* and that $p$ is a *bound for a recursive trace for every* $f \leq_T A$.

The first result on reals that are low for Schnorr tests comes from Terwijn and Zambella [15]. Later, Kjos-Hanssen, Nies, and Stephan used a similar technique to demonstrate that, given Terwijn and Zambella's result, the reals that are low for Schnorr tests are precisely the reals that are low for Schnorr randomness [9].

**Theorem 1.11.** [15] *A set is recursively traceable if and only if it is low for Schnorr tests.*

**Theorem 1.12.** [9] *A set is recursively traceable if and only if it is low for Schnorr randomness.*

This will allow us to refer to these reals as simply "low for Schnorr" or "Schnorr low" and to the property as "lowness for Schnorr."

Although the reals that are low for Martin-Löf randomness are precisely those that are $K$-trivial (that is, trivial in the sense of Martin-Löf), being low for Schnorr is not equivalent to being Schnorr trivial. All reals that are low for Schnorr are hyperimmune free, and there is a Turing complete Schnorr trivial. This Schnorr trivial is clearly not hyperimmune free and is thus not Schnorr low. The best that can be hoped for is that all Schnorr lows are Schnorr trivial. We show that this is, in fact, the case, and we will prove an even stronger theorem.

## 2. The main result

**Theorem 2.1.** *A real $A$ is Schnorr low if and only if it is Schnorr trivial and hyperimmune free.*

The first part of the proof of this theorem is very much like the argument in the proof of Theorem 9 in [6], while the other part relies on the totality of Turing reductions within the hyperimmune-free degrees to construct a computable Turing machine.

Before we prove the main theorem, we state the following result which we will need from [1].

**Theorem 2.2** (Kraft-Chaitin Theorem [1])**.** *Let $\langle d_i, \sigma_i\rangle_{i\in\omega}$ be a recursive sequence with $d_i \in \omega$ and $\sigma_i \in 2^{<\omega}$ for all $i$ such that $\sum_i \frac{1}{2^{d_i}} \leq 1$. (Such a sequence is called a* Kraft-Chaitin set*, and each element of the sequence is called a* Kraft-Chaitin axiom*.) Then there are strings $\tau_i$ and a prefix-free machine $M$ such that $dom(M) = \{\tau_i \mid i \in \omega\}$ and for all $i$ and $j$ in $\omega$,*

(1) *if $i \neq j$, then $\tau_i \neq \tau_j$,*
(2) *$|\tau_i| = d_i$,*
(3) *and $M(\tau_i) = \sigma_i$.*

The Kraft-Chaitin Theorem allows us to construct a prefix-free Turing machine by specifying only the lengths of the strings in the domain rather than the strings themselves. We will therefore identify $\langle \tau, \sigma\rangle$ with $\langle d, \sigma\rangle$, where $d = |\tau|$, throughout.

*Proof of Theorem 2.1.* Let $A \in 2^\omega$ be low for Schnorr. Since we know that $A$ must be hyperimmune free, we need only show that the following is true.

$$(\forall M \text{ comp.})\,(\exists M' \text{ comp.})\,(\exists c \in \omega)\,(\forall n \in \omega)\,[K_{M'}(A\upharpoonright n) \leq K_M(0^n) + c]$$

We consider an arbitrary computable machine $M$ and fix an enumeration $\langle M_s\rangle_{s\in\omega}$. As per [5], without loss of generality, we may assume that $\mu(M) = 1$. We must build a computable machine $M'$ and determine a constant $c$ so that $M'$ and $c$ witness the Schnorr triviality of $A$.

Our construction will proceed in stages. However, we will need to consider two different types of stages: those in the enumeration of the elements of $M$, and the stages of our construction. For clarity, we will refer to the former as $M$-stages and to the latter simply as stages.

We now define two recursive sequences of natural numbers to aid us in our construction: $\langle s_k\rangle_{k\in\omega}$ and $\langle n_k\rangle_{k\in\omega}$, which we define as follows.

$$\begin{aligned} s_k &= \min\left\{s \mid \mu(M_s) \geq 1 - \frac{1}{2^k}\right\} \\ n_k &= \max\{|\sigma| \mid \sigma \in M_{s_k}\} \end{aligned}$$

Note that since $\mu(M) = 1$, both $s_k$ and $n_k$ will be defined for all $k$.

Since $A$ is Schnorr low and thus recursively traceable, we know that there is an order function $p : \omega \longrightarrow \omega$ such that

$$(\forall f \leq_T A)\,(\exists r : \omega \longrightarrow \omega \text{ rec.})\,(\forall n \in \omega)\left[f(n) \in D_{r(n)} \text{ and } |D_{r(n)}| \leq p(n)\right].$$

Terwijn and Zambella showed that if this statement holds for one such function $p$, it holds for any order function [15]. Therefore, we may choose the function $p$ such that $p(0) = 1$ and $p(k) = k$ for all $k > 0$. We let $f(k) = A\upharpoonright n_k$. Since $\langle n_k\rangle_{k\in\omega}$ is a recursive sequence, $f \leq_T A$.

**Stage $k = 0$:** We have $p(0) = 1$. Since $A\upharpoonright n_0$ is the empty sequence and no axioms have entered $M$, we set $M'_0 = \emptyset$.

**Stage $k = 1$:** We have $p(1) = 1$ and $f(1) = A\upharpoonright n_1$. Therefore, $A\upharpoonright n_1 \in D_{r(1)}$ and $|D_{r(1)}| \leq 1$, so we can identify the first $n_1$ bits of $A$. If $\langle d, \tau\rangle$ enters $M$ at $M$-stage $s \leq s_1$, we add $\langle d+1, A\upharpoonright|\tau|\rangle$ to $M'$. We define $M'_1$ to be the sequence of Kraft-Chaitin axioms that have been enumerated into $M'$ by the end of this stage.

**Stage $k > 1$:** In this case, $p(k) = k$, and $f(k) = A\upharpoonright n_k$ must belong to the $\leq k$ elements of $D_{r(k)}$. Therefore, without loss of generality, we can represent $D_{r(k)}$ as $\{\sigma_1, \ldots, \sigma_k\}$. We are now able to further limit the elements of $D_{r(k)}$ that we must actually consider.

We call an element of $D_{r(k)}$ *acceptable* if we cannot rule out the possibility that it is $A\upharpoonright n_k$ using its length and its compatibility with strings previously identified as acceptable as criteria. Clearly, only those elements of $D_{r(k)}$ that have length $n_k$ can be acceptable. Furthermore, since $A\upharpoonright n_k$ extends $A\upharpoonright n_{k-1}$, an acceptable element of $D_{r(k)}$ must extend an acceptable element of $D_{r(k-1)}$. These acceptable strings can be recognized recursively, and $A\upharpoonright n_k$ will clearly be among them. Suppose there are $l$ acceptable strings. Clearly, $l$ must be less than or equal to $k$.

Now we may enumerate axioms into $M'$. If $\langle d, \tau\rangle$ enters $M$ at $M$-stage $s$ for $s_{k-1} < s \leq s_k$, we add $\langle d+1, \sigma_i\rangle$ to $M'$ at stage $s$ for each acceptable string $\sigma_i$ for a total of $l$ new axioms. If $l$ is strictly less than $k$, we enumerate the axiom $\langle d+1, \sigma_i\rangle$ into $M'$ $k-l$ times, where $i$ is the smallest index of an acceptable string. This gives us a total of $l+(k-l)=k$ new axioms for each such $\langle d, \tau\rangle$. We let $M'_s$ be the sequence of Kraft-Chaitin axioms that have been enumerated into $M'$ by the end of stage $s$.

Now we define $M'$ to be $\cup_s M'_s$.

**Lemma 2.3.** *$M'$ is a Kraft-Chaitin set.*

*Proof.* We first note that, since our construction is recursive, $M'$ is r.e.

Now we must show that $\mu(M') \leq 1$. We know that $\mu(M) = 1$. Suppose we divide the measure of $M$ into a sequence of intervals $\langle I_k\rangle_{k>0}$, where for each $k$, we define $I_k = (1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}]$. Note that $I_k$ has measure precisely $\frac{1}{2^k}$ for each $k$. We would like to show that for each $I_k$, no more than $\frac{k}{2^{k+1}}$ enters $M'$. Theoretically, precisely the axioms that enter $M$ at a stage $s$ such that $s_{k-1} < s \leq s_k$ should contribute to $I_k$. However, it is possible that this will not be the case, since $s_{k-1}$ is defined as the least $M$-stage $s$ such that $\mu(M_s) \geq 1 - \frac{1}{2^{k-1}}$ and not the least stage $s$ such that $\mu(M_s) = 1 - \frac{1}{2^{k-1}}$. Therefore, it may be that $\mu(M_{s_{k-1}}) > 1 - \frac{1}{2^{k-1}}$, and the most we can say is that the axioms that contribute to $I_k$ enter $M$ at or before $M$-stage $s_k$.

Suppose $n \leq k$, and suppose $\langle d, \sigma\rangle$ enters $M$ at an $M$-stage $s$ such that $s_{n-1} < s \leq s_n$. There are now four possibilities: that $\mu(M_s) < 1 - \frac{1}{2^{k-1}}$ and $\langle d, \sigma\rangle$ contributes nothing to $I_k$, that $\mu(M_{s-1}) < 1 - \frac{1}{2^{k-1}} < \mu(M_s)$, that $1 - \frac{1}{2^{k-1}} < \mu(M_{s-1}) < \mu(M_s) < 1 - \frac{1}{2^k}$, or that $1 - \frac{1}{2^{k-1}} < \mu(M_{s-1}) < 1 - \frac{1}{2^k}$ and $\mu(M_s) > 1 - \frac{1}{2^k}$.

In the first case, $\langle d, \sigma\rangle$ contributes nothing to $I_k$, so we need not consider it. In the third case, all of the measure that $\langle d, \sigma\rangle$ contributes to $\mu(M)$ is contained in $I_k$, so $n = k$ and $\frac{k}{2^{d+1}}$ enters $\mu(M')$ when $\frac{1}{2^d}$ enters $\mu(M)$.

The second and fourth cases are analogous. In the second case, some of the $\frac{1}{2^d}$ that $\langle d, \sigma\rangle$ contributes to $\mu(M)$ is contained in $I_{k-1}$. Suppose that $0 < q < 1$ is the fraction of this $\frac{1}{2^d}$ that is actually in $I_k$. Since $\langle d, \sigma\rangle$ is dealt with in our construction at stage $n$, $\frac{n}{2^{d+1}}$ will enter $\mu(M')$. We assign the proportional amount of measure $q \cdot \frac{n}{2^{d+1}}$ to the part of $\mu(M')$ corresponding to $I_k$, so for the $q \cdot \frac{1}{2^d}$ entering $I_k$, $q \cdot \frac{n}{2^{d+1}} \leq q \cdot \frac{k}{2^{d+1}}$ enters $\mu(M')$.

In the fourth case, some of the measure contributed to $\mu(M)$ by $\langle d, \sigma\rangle$ is contained in $I_k$ and some is contained in $I_m$ for $m > k$. We treat this precisely as we did in the second case and consider only the portion of this measure that is contained in $I_k$.

In each of these cases, when $\langle d, \sigma\rangle$ enters $M$ and contributes some measure to $I_k$, no more than $\frac{k}{2}$ of that measure enters $\mu(M')$. After summing up the contributions to the measure of $M'$ corresponding to each $I_k$, we can see that for each interval $I_k$, no more than $\frac{k}{2} \cdot \frac{1}{2^k} = \frac{k}{2^{k+1}}$ is added to $\mu(M')$. This allows us to see that $\mu(M') \leq \sum_{k\in\omega} \frac{k}{2^{k+1}} = 1$, so $M'$ is a Kraft-Chaitin set.

□

**Lemma 2.4.** *$\mu(M')$ is a recursive real.*

*Proof.* It is enough to show that $\mu(M')$ is the limit of a recursive sequence of rationals $\langle q_k\rangle_{k\in\omega}$ such that there is a recursive function $f$ such that for all $k$, $|\mu(M') - q_{f(k)}| < \frac{1}{2^k}$.

We begin by observing that since $\mu(M'_k)$ is a finite sum of rationals for each $k$, each $\mu(M'_k)$ is a rational. Furthermore, by our construction, $\langle \mu(M'_k)\rangle_{k\in\omega}$ is a recursive sequence, and $\mu(M') = \lim_k \mu(M'_k)$, so we may take our recursive sequence of rationals to be $\langle \mu(M'_k)\rangle_{k\in\omega}$.

Finally, after stage $k$, no more than $\frac{1}{2^k}$ will enter $M$, so, as demonstrated in the proof of Lemma 2.3, no more than $\sum_{j>k} \frac{j}{2^{j+1}}$ can be added to $\mathrm{dom}(M')$. We can therefore see that

$$|\mu(M') - \mu(M'_k)| \leq \sum_{j>k} \frac{j}{2^{j+1}} = \frac{k+2}{2^{k+1}}$$

for each $k$. This inequality will allow us to find a recursive function $f$ as mentioned in the first paragraph, so we can see that $\mu(M')$ is a recursive real. □

By these two lemmas and the Kraft-Chaitin Theorem, we may consider $M'$ to be a computable Turing machine.

**Lemma 2.5.** *$M'$ and the constant 1 witness the Schnorr triviality of $A$ with respect to $M$.*

*Proof.* Let $n \in \omega$ be arbitrary. We would like to show that $K_{M'}(A\restriction n) \leq K_M(0^n) + 1$. If $K_M(0^n) = \infty$, we are done. Otherwise, suppose that $K_M(0^n) = d$ for some $d \in \omega$. During our construction, we would have added $\langle d+1, \tau\rangle$ to $M'$ for all $\tau$ of length $n$ that were substrings of acceptable strings at that point. Since $A\restriction n$ is a substring of $A\restriction n_k$ for any $k$ such that $n_k \geq n$, we can see that $K_{M'}(A\restriction n) \leq d+1 = K_M(0^n)+1$, so $M'$ and the constant 1 witness the Schnorr triviality of $A$ with respect to $M$. □

Therefore, for any arbitrary computable Turing machine $M$, we may produce another computable Turing machine $M'$ and a constant $c$ witnessing the Schnorr triviality of $A$ with respect to $M$, so $A$ is Schnorr trivial.

We now turn our attention to the reals which are Schnorr trivial and hyperimmune free to complete the proof of this theorem. We will assume that a real $A$ is hyperimmune free but not Schnorr low and proceed to show that $A$ cannot be Schnorr trivial.

Since $A$ is not Schnorr low, then for any order function $p : \omega \longrightarrow \omega$,

$$\left(\exists g^A\right) (\forall r : \omega \longrightarrow \omega \text{ rec.}) (\exists n \in \omega) \left[g^A(n) \notin D_{r(n)} \text{ or } |D_{r(n)}| > p(n)\right].$$

By the Padding Lemma, it will be enough to define a computable $M$ such that

$$(\forall M_i \text{ comp.}) (\exists n \in \omega) \left[K_{M_i}(A\restriction n) > K_M(0^n) + i\right].$$

Now we define $l(n)$ to be the use function for $g$ relative to all possible $A$. We note that since we are working within the hyperimmune-free degrees, all Turing reductions are total. Therefore, $l(n)$ will be a total recursive function. We then use this function to define $M$ as follows.

$$M = \{\langle n+1, 0^{l(n)}\rangle \mid n \in \omega\}$$

We note that $\mu(M) = 1$ and that $M$ is a Kraft-Chaitin set, so we may treat $M$ as a computable Turing machine.

We now select $p(n) = 2^{2n+1}$, so there is a $g^A$ such that

$$(\forall r : \omega \longrightarrow \omega \text{ rec.}) (\exists n \in \omega) \left[g^A(n) \notin D_{r(n)} \text{ or } |D_{r(n)}| > 2^{2n+1}\right].$$

Now we note that for any $M_i$, there are at most $2^{2n+1}$ many $\sigma \in 2^{<\omega}$ of length $l(n)$ such that $K_{M_i}(\sigma) \leq 2n+1$. We use this information about the $M_i$s to define functions $r_i$ such that

$$D_{r_i(n)} = \{g^\sigma(n) \mid K_{M_i}(\sigma) \leq 2n+1\}.$$

If $M_i$ is computable, $D_{r_i(n)}$ can be determined recursively by considering the Schnorr test associated with $M_i$ to find the appropriate $\sigma$s. Therefore, in this case, $r_i$ will be a recursive function. For such an $r_i$,

$$(\exists n \in \omega) \left[g^A(n) \notin D_{r_i(n)} \text{ or } |D_{r_i(n)}| > 2^{2n+1}\right].$$

We know from the definition of $r_i$ that $|D_{r_i(n)}| \leq 2^{2n+1}$ for all $n$, so we can deduce that $(\exists n \in \omega)\left[g^A(n) \notin D_{r_i(n)}\right]$. Therefore,

$$K_{M_i}(A \upharpoonright l(n)) > 2n + 1 = K_M(0^{l(n)}) + n.$$

This is enough to show that $K_{M_i}(A \upharpoonright l(n)) > K_M(0^{l(n)}) + i$ for any computable $M_i$. If $n \geq i$, we have $K_{M_i}(A \upharpoonright l(n)) > K_M(0^{l(n)}) + i$ and we are done. There are only finitely many $n$ such that $n < i$, so in that case, there is another recursive function $r'_i$ that will produce a $D_{r'_i(n)}$ demonstrating that $K_{M_i}(A \upharpoonright l(n)) > K_M(0^{l(n)}) + i$ for these $n$ as well. Therefore, $A$ is not Schnorr trivial, and we have our contradiction.

□

## 3. Conclusions

In the Martin-Löf case, all of the triviality and lowness notions coincide. Although this does not happen in the Schnorr case, it makes it natural to ask if there is a subset of the Turing degrees in which it does. We have answered this question positively in this paper: the hyperimmune-free degrees are the largest possible such subset. Now we may ask an additional question: is there any significance to the fact that it is the hyperimmune-free degrees with this property?

We note that the hyperimmune-free Turing degrees are precisely those that contain precisely one *tt*-degree and one *wtt*-degree [8]. Since this is where Schnorr triviality and lowness for Schnorr randomness coincide, Theorem 2.1 suggests that the *wtt*-degrees or the *tt*-degrees may be a more appropriate structure in which to consider Schnorr triviality.

Furthermore, the Schnorr trivial reals do not behave as one might expect in the Turing degrees: they are not downward closed [3], and they may have arbitrarily high Turing degree [6]. These results also suggest, when put together, that the Schnorr trivial reals may be more properly considered in another degree structure.

Downey, Griffiths, and Laforte connected Schnorr trivial reals to the *tt*-degrees in [3], when they proved that the Schnorr trivial reals are downward closed in the *tt*-degrees. In [7], we show that this is not the case in the *wtt*-degrees. Since it seems desirable for the Schnorr trivial reals to be closed downward, this suggests that the *tt*-degrees are a more appropriate structure than the *wtt*-degrees. We also show in [7] that Schnorr triviality is equivalent to a number of other properties closely tied to *tt*-reducibility. One of these properties allows us to see easily that the Schnorr trivial reals are actually closed under join and therefore form an ideal in the *tt*-degrees, answering a question of Downey, Griffiths, and Laforte in [3]. The proof of this result is based on a characterization of Schnorr triviality that is very similar to recursive traceability.

We also note that the Schnorr triviality of a real is determined entirely by computable Turing machines and *tt*-reducibility is based on the notion of total Turing functionals. These factors all lead us to believe that the most appropriate setting in which to analyze the Schnorr trivial reals is the *tt*-degrees.

It would also be interesting to study the Turing degrees that contain Schnorr trivials more closely. For instance, the following question may prove useful.

**Question 3.1.** If a hyperimmune Turing degree contains a Schnorr trivial real, must it contain at least two *tt*-degrees that contain Schnorr trivial reals? Infinitely many? An antichain of such *tt*-degrees?

We may also consider the following question mentioned in [10], which is still open.

**Question 3.2.** Is the class of $K$-trivial reals definable in the $\Delta^0_2$ degrees or the recursively enumerable degrees?

We may ask similar questions about Schnorr trivial and Schnorr low reals. However, we cannot limit ourselves naturally to a given type of degree in the case of Schnorr triviality, though it is possible for Schnorr lowness.

**Question 3.3.** Is the class of Schnorr trivial reals definable in the Turing degrees or *tt*-degrees?

**Question 3.4.** Is the class of Schnorr low reals definable in the hyperimmune-free degrees?

Of course, the answers to these questions are not known for the $K$-trivial reals, either, so there will be no direct comparison. However, if we find that the Schnorr trivial reals are definable in one degree structure but not the other, that might provide evidence that one is more natural than the other.

If the natural structure for the Schnorr trivial reals is the $tt$-degrees, we should consider the possibility that this may be the natural structure for the Schnorr random reals as well. Nies, Stephan, and Terwijn have proven that Martin-Löf randomness and Schnorr randomness are separable in the high degrees [12]. As the Turing and $tt$-degrees of the Schnorr random reals are investigated further, it is possible that their behavior in the $tt$-degrees will prove interesting.

We may approach this by asking whether Schnorr random reals have the same properties in the Turing degrees as the Martin-Löf random reals do. If they do not, we may then ask whether the Schnorr random reals would have these properties in the $tt$-degrees. We may again consider definability, though once again, the question, proposed by Kučera, of the definability of the Martin-Löf random reals in the Turing degrees is still open [10].

**Question 3.5.** Is the class of Schnorr random reals definable in the Turing degrees or the $tt$-degrees?

One theorem concerning Martin-Löf random reals that should certainly be considered in the context of Schnorr randomness is Demuth's Theorem.

**Theorem 3.6.** [2] *If $A$ is a Martin-Löf random real and $0 <_T B \leq_{tt} A$, then there is a Martin-Löf random set $C$ such that $C \equiv_{wtt} B$.*

This theorem is interesting for several reasons. First, it shows that in some sense, the Martin-Löf random reals are downward closed. Second, its dependence on the use of different reducibilities makes it ideal for our purposes in understanding the role that the different reducibilities play in randomness. This leads to the following questions.

**Question 3.7.** Can Demuth's Theorem be sharpened? For instance, can we conclude instead that there is a Martin-Löf random $C \equiv_{tt} B$?

**Question 3.8.** Does Demuth's Theorem hold for Schnorr randomness?

## References


[1] Gregory J. Chaitin. A theory of program size formally identical to information theory. *J. Assoc. Comput. Mach.*, 22:329–340, 1975.

[2] Osvald Demuth. Remarks on the structure of tt-degrees based on constructive measure theory. *Comment. Math. Univ. Carolin.*, 29(2):233–247, 1988.

[3] Rod Downey, Evan Griffiths, and Geoffrey Laforte. On Schnorr and computable randomness, martingales, and machines. *Math. Log. Q.*, 50(6):613–627, 2004.

[4] Rod Downey, Denis R. Hirschfeldt, André Nies, and Sebastiaan A. Terwijn. Calibrating randomness. *Bull. Symbolic Logic*, 12(3):411–491, 2006.

[5] Rodney G. Downey and Evan J. Griffiths. Schnorr randomness. *J. Symbolic Logic*, 69(2):533–554, 2004.

[6] Johanna N. Y. Franklin. Schnorr trivial reals: A construction. *Arch. Math. Logic*. To appear.

[7] Johanna N. Y. Franklin and Frank Stephan. Schnorr trivial sets and truth-table reducibility. In preparation.

[8] Carl G. Jockusch, Jr. Relationships between reducibilities. *Trans. Amer. Math. Soc.*, 142:229–237, 1969.

[9] Bjørn Kjos-Hanssen, André Nies, and Frank Stephan. Lowness for the class of Schnorr random reals. *SIAM J. Comput.*, 35(3):647–657 (electronic), 2005.

[10] Joseph S. Miller and André Nies. Randomness and computability: Open questions. *Bull. Symbolic Logic*, 12(3):390–410, 2006.

[11] André Nies. Lowness properties and randomness. *Adv. Math.*, 197(1):274–305, 2005.

[12] André Nies, Frank Stephan, and Sebastiaan A. Terwijn. Randomness, relativization and Turing degrees. *J. Symbolic Logic*, 70(2):515–535, 2005.

[13] C.-P. Schnorr. A unified approach to the definition of random sequences. *Math. Systems Theory*, 5:246–258, 1971.

[14] Robert I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic. Springer-Verlag, 1987.
[15] Sebastiaan A. Terwijn and Domenico Zambella. Computational randomness and lowness. *J. Symbolic Logic*, 66(3):1199–1205, 2001.

Department of Mathematics, National University of Singapore, 2, Science Drive 2, Singapore 117543, Singapore
*E-mail address*: franklin@math.nus.edu.sg